Newton's method in the Mandelbrot set

Claude Heiland-Allen claude@mathr.co.uk

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Abstract

The Mandelbrot set's mu-atoms each have a nucleus. Each mu-atom has adjacent children, connected by bond points. Given an estimate of the nucleus, and its period, one can compute the nucleus using Newton's method in one variable. Given the nucleus and its period, one can compute bond points using Newton's method in two variables. The required derivatives can be computed efficiently using recurrence relations.

1 Prerequisites

Definition 1 (Roots of functions)**.** *A root of a function f is an x such that $f(x) = 0$.*

Definition 2 (Jacobian matrix)**.** *The Jacobian matrix is the matrix of all first-order partial derivatives of a vector- or scalar-valued function with respect to another vector:*

$$\boldsymbol{y} = f(\boldsymbol{x}) \qquad J_f = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix}$$

Definition 3 (Newton's method in one variable)**.** *Given a well-behaved function f and a first guess x_0 for a root of f, better approximations can be found by:*

$$x_{m+1} = x_m - \frac{f(x_m)}{\frac{\partial}{\partial x} f(x_m)}$$

Definition 4 (Newton's method in more than one variable)**.** *Given a well-behaved function f and a first guess $\boldsymbol{v}_0$ for a root of f, better approximations can be found by:*

$$\boldsymbol{v}_{m+1} = \boldsymbol{v}_m - J_f(\boldsymbol{v}_m)^{-1} f(\boldsymbol{v}_m)$$

or equivalently by solving:

$$J_f(\boldsymbol{v}_m)(\boldsymbol{v}_{m+1} - \boldsymbol{v}_m) = -f(\boldsymbol{v}_m)$$

2 Finding features

Definition 5 (Quadratic function)**.**

$$f(z, c) = z^2 + c$$

Definition 6 (Iterated function)**.**

$$\begin{aligned} f^0(z, c) &= z \\ f^{m+1}(z, c) &= f(f^m(z, c), c) \end{aligned}$$

Theorem 1 (Finding nucleus)**.** *A nucleus n of period p satisfies $f^p(0, n) = 0$. Applying Newton's method in one variable:*

$$n_{m+1} = n_m - \frac{f^p(0, n_m)}{\frac{\partial}{\partial c} f^p(0, n_m)}$$

Theorem 2 (Finding bond points)**.** *Given a parent nucleus n of period p, the bond point b at angle a (measured in turns) satisfies:*

$$\begin{aligned} f^p(w, b) - w &= 0 \\ \frac{\partial}{\partial z} f^p(w, b) - e^{2\pi i a} &= 0 \end{aligned}$$

Defining:

$$g \begin{pmatrix} z \\ c \end{pmatrix} = \begin{pmatrix} f^p(z, c) - z \\ \frac{\partial}{\partial z} f^p(z, c) - e^{2\pi i a} \end{pmatrix}$$

and applying Newton's method in two variables:

$$\begin{aligned} v_0 &= \begin{pmatrix} n \\ n \end{pmatrix} \\ J_g(\boldsymbol{v}_m)(\boldsymbol{v}_{m+1} - \boldsymbol{v}_m) &= -g(\boldsymbol{v}_m) \end{aligned}$$

where

$$J_g = \begin{pmatrix} \frac{\partial}{\partial z}(f^p(z, c) - z) & \frac{\partial}{\partial c}(f^p(z, c) - z) \\ \frac{\partial}{\partial z}(\frac{\partial}{\partial z} f^p(z, c) - e^{2\pi i a}) & \frac{\partial}{\partial c}(\frac{\partial}{\partial z} f^p(z, c) - e^{2\pi i a}) \end{pmatrix}$$

3 Recurrence relations

Definition 7 (Naming derivatives)**.** *Picking names arbitrarily for the required derivatives:*

$$\begin{aligned} A_m &= f^m \\ B_m &= \frac{\partial}{\partial z} f^m \\ C_m &= \frac{\partial}{\partial z} \frac{\partial}{\partial z} f^m \\ D_m &= \frac{\partial}{\partial c} f^m \\ E_m &= \frac{\partial}{\partial c} \frac{\partial}{\partial z} f^m \end{aligned}$$

Theorem 3 (Nucleus recurrence)**.** *Theorem 1 can now be rewritten as:*

$$n_{m+1} = n_m - \frac{A_p(0, n_m)}{D_p(0, n_m)}$$

Theorem 4 (Bond point recurrence)**.** *Theorem 2 can now be rewritten as:*

$$\begin{pmatrix} w_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} n \\ n \end{pmatrix}$$

$$\begin{pmatrix} B_p - 1 & D_p \\ C_p & E_p \end{pmatrix} \begin{pmatrix} w_{m+1} - w_m \\ b_{m+1} - b_m \end{pmatrix} = - \begin{pmatrix} A_p - w_m \\ B_p - e^{2\pi i a} \end{pmatrix}$$

where $\{A, B, C, D, E\}_p$ *are evaluated at* (w_m, b_m)*.*

Theorem 5 (Initial values)**.**

$$\begin{aligned} A_0 &= z \\ B_0 &= 1 \\ C_0 &= 0 \\ D_0 &= 0 \\ E_0 &= 0 \end{aligned}$$

Theorem 6 (Recurrence relation)**.**

$$\begin{aligned} A_{m+1} &= A_m^2 + c \\ B_{m+1} &= 2 A_m B_m \\ C_{m+1} &= 2(B_m^2 + A_m C_m) \\ D_{m+1} &= 2 A_m D_m + 1 \\ E_{m+1} &= 2(A_m E_m + B_m D_m) \end{aligned}$$

4 Implementation

Example implementation pseudo-code in Haskell:

```
-- compute successive approximations to nucleus
nucleus :: N -> C -> [C]
nucleus p n0 = iterate go n0
  where
    go nm =
      let (a, _, _, d, _) = derivatives p 0 nm
      in  nm - a / d

-- compute successive approximations to bond point
bond :: N -> C -> R -> [C]
bond p n0 a0 = map snd (iterate go (n0, n0))
  where
    t = cis (2 * pi * a0)
    go (wm, bm) =
      let (a, b, c, d, e) = derivatives p wm bm
          jm = toMatrix ((b - 1, d), (c, e))
          xm = toVector (wm - a, t - b)
          dxm = solve jm xm
          xm1 = xm + dxm
          (wm1, bm1) = fromVector xm1
      in  (wm1, bm1)

-- compute derivatives via recurrence relations
derivatives :: N -> C -> C -> (C, C, C, C, C)
derivatives p z0 c0 =
    iterate go (z0, 1, 0, 0, 0) `genericIndex` p
  where
    go (a, b, c, d, e) =
      ( a^2 + c0
      , 2 * a * b
      , 2 * (b^2 + a * c)
      , 2 * a * d + 1
      , 2 * (a * e + b * d)
      )
```

5 Proofs

Derivation of A_0.

$$
\begin{aligned}
& A_0 \\
= & \qquad \{ \text{ definition of } A \text{ } \} \\
& f^0(z, c) \\
= & \qquad \{ \text{ definition of } f \text{ } \} \\
& z
\end{aligned}
$$

□

Derivation of B_0.

$$
\begin{aligned}
& B_0 \\
= & \qquad \{ \text{ definition of } B \text{ } \} \\
& \frac{\partial}{\partial z} f^0(z, c) \\
= & \qquad \{ \text{ definition of } f \text{ } \} \\
& \frac{\partial}{\partial z} z \\
= & \qquad \{ \text{ derivative } \} \\
& 1
\end{aligned}
$$

□

Derivation of C_0.

$$
\begin{aligned}
& C_0 \\
= & \qquad \{ \text{ definition of } C \text{ } \} \\
& \frac{\partial}{\partial z} \frac{\partial}{\partial z} f^0(z, c) \\
= & \qquad \{ \text{ definition of } f \text{ } \} \\
& \frac{\partial}{\partial z} \frac{\partial}{\partial z} z \\
= & \qquad \{ \text{ derivative } \} \\
& \frac{\partial}{\partial z} 1 \\
= & \qquad \{ \text{ derivative } \} \\
& 0
\end{aligned}
$$

□

Derivation of D_0.

$$
\begin{aligned}
& D_0 \\
= & \quad \{ \text{ definition of } D \text{ } \} \\
& \frac{\partial}{\partial c} f^0(z, c) \\
= & \quad \{ \text{ definition of } f \text{ } \} \\
& \frac{\partial}{\partial c} z \\
= & \quad \{ \text{ derivative } \} \\
& 0
\end{aligned}
$$

□

Derivation of E_0.

$$
\begin{aligned}
& E_0 \\
= & \quad \{ \text{ definition of } E \text{ } \} \\
& \frac{\partial}{\partial c} \frac{\partial}{\partial z} f^0(z, c) \\
= & \quad \{ \text{ definition of } f \text{ } \} \\
& \frac{\partial}{\partial c} \frac{\partial}{\partial z} z \\
= & \quad \{ \text{ derivative } \} \\
& \frac{\partial}{\partial c} 1 \\
= & \quad \{ \text{ derivative } \} \\
& 0
\end{aligned}
$$

□

Derivation of A_{m+1}.

$$
\begin{aligned}
& A_{m+1} \\
= & \quad \{ \text{ definition of } A \text{ } \} \\
& f^{m+1}(z, c) \\
= & \quad \{ \text{ definition of } f \text{ } \} \\
& f(f^m(z, c), c) \\
= & \quad \{ \text{ definition of } A \text{ } \} \\
& f(A_m, c) \\
= & \quad \{ \text{ definition of } f \text{ } \} \\
& A_m^2 + c
\end{aligned}
$$

□

Derivation of B_{m+1}.

$$
\begin{aligned}
& B_{m+1} \\
= & \qquad \{ \text{ definition of } B \text{ } \} \\
& \frac{\partial}{\partial z} A_{m+1} \\
= & \qquad \{ \text{ definition of } A \text{ } \} \\
& \frac{\partial}{\partial z} (A_m^2 + c) \\
= & \qquad \{ \text{ distributivity } \} \\
& \frac{\partial}{\partial z} A_m^2 + \frac{\partial}{\partial z} c \\
= & \qquad \{ \text{ constant derivative } \} \\
& \frac{\partial}{\partial z} A_m^2 + 0 \\
= & \qquad \{ \text{ zero } \} \\
& \frac{\partial}{\partial z} A_m^2 \\
= & \qquad \{ \text{ chain rule } \} \\
& 2A_m (\frac{\partial}{\partial z} A_m) \\
= & \qquad \{ \text{ definition of } B \text{ } \} \\
& 2A_m B_m
\end{aligned}
$$

□

Derivation of C_{m+1}.

$$
\begin{aligned}
& C_{m+1} \\
= & \qquad \{ \text{ definition of } C \text{ } \} \\
& \frac{\partial}{\partial z} B_{m+1} \\
= & \qquad \{ \text{ definition of } B \text{ } \} \\
& \frac{\partial}{\partial z}(2A_m B_m) \\
= & \qquad \{ \text{ linearity } \} \\
& 2\frac{\partial}{\partial z}(A_m B_m) \\
= & \qquad \{ \text{ product rule } \} \\
& 2((\frac{\partial}{\partial z}A_m)B_m + A_m(\frac{\partial}{\partial z}B_m)) \\
= & \qquad \{ \text{ definition of } B \text{ } \} \\
& 2(B_m B_m + A_m(\frac{\partial}{\partial z}B_m)) \\
= & \qquad \{ \text{ algebra } \} \\
& 2(B_m^2 + A_m(\frac{\partial}{\partial z}B_m)) \\
= & \qquad \{ \text{ definition of } C \text{ } \} \\
& 2(B_m^2 + A_m C_m)
\end{aligned}
$$

□

Derivation of D_{m+1}.

$$
\begin{aligned}
& D_{m+1} \\
= & \qquad \{ \text{ definition of } D \ \} \\
& \frac{\partial}{\partial c} A_{m+1} \\
= & \qquad \{ \text{ definition of } A \ \} \\
& \frac{\partial}{\partial c} (A_m^2 + c) \\
= & \qquad \{ \text{ distributivity } \} \\
& \frac{\partial}{\partial c} A_m^2 + \frac{\partial}{\partial c} c \\
= & \qquad \{ \text{ derivative } \} \\
& \frac{\partial}{\partial c} A_m^2 + 1 \\
= & \qquad \{ \text{ chain rule } \} \\
& 2A_m (\frac{\partial}{\partial c} A_m) + 1 \\
= & \qquad \{ \text{ definition of } D \ \} \\
& 2A_m D_m + 1
\end{aligned}
$$

□

Derivation of E_{m+1}.

$$
\begin{aligned}
& E_{m+1} \\
= & \qquad \{ \text{ definition of } E \ \} \\
& \frac{\partial}{\partial c} B_{m+1} \\
= & \qquad \{ \text{ definition of } B \ \} \\
& \frac{\partial}{\partial c} (2A_m B_m) \\
= & \qquad \{ \text{ linearity } \} \\
& 2 \frac{\partial}{\partial c} (A_m B_m) \\
= & \qquad \{ \text{ product rule } \} \\
& 2(A_m (\frac{\partial}{\partial c} B_m) + (\frac{\partial}{\partial c} A_m) B_m) \\
= & \qquad \{ \text{ definition of } E \ \} \\
& 2(A_m E_m + (\frac{\partial}{\partial c} A_m) B_m) \\
= & \qquad \{ \text{ definition of } D \ \} \\
& 2(A_m E_m + D_m B_m)
\end{aligned}
$$

□