Fractal Dimension of Julia Sets

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Fractal Dimension

How Long is a Coast?

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How long is a coast?

How Long is a Coast?

It looks this long.

How Long is a Coast?

But as you look closer,

How Long is a Coast?

more details appear,

How Long is a Coast?

giving a longer length,

How Long is a Coast?

so when do you stop?

How Long is a Coast?

How long is a coast?

How Long is a Coast?

It gets longer the closer you look.

How Long is a Coast?

The concept of "length" is usually meaningless for geographical curves. They can be considered superpositions of features of widely scattered characteristic sizes; ***as even finer features are taken into account, the total measured length increases****, and there is usually no clear-cut gap or crossover, between the realm of geography and details with which geography need not be concerned.*

– B. B. Mandelbrot
"How long is the coast of Britain?"
Science: 156, 1967, 636-638

How Long is a Coast?

A better question:

How Long is a Coast?

A better question:
How much longer does a coast get the closer you look?

Box-Counting Dimension

The idea:

Box-Counting Dimension

The idea:

- Pick a box size r.

Box-Counting Dimension

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- Pick a box size r.
- Cover the boundary with boxes of size r.

Box-Counting Dimension

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- Count how many boxes are needed N_r.

Box-Counting Dimension

The idea:

- Pick a box size r.
- Cover the boundary with boxes of size r.
- Count how many boxes are needed N_r.
- See how quickly N_r increases as r gets smaller.

Box-Counting Dimension

The formal definition:

$$\dim = \lim_{r \to 0} -\frac{\log N_r}{\log r}$$

Box-Counting Dimension

The formal definition:

$$\dim = \lim_{r \to 0} - \frac{\log N_r}{\log r}$$

Converges very slowly, not practical.

Box-Counting Dimension

A more practical definition:

$$\dim = \lim_{r \to 0} \log_2 \frac{N_r}{N_{2r}}$$

Box-Counting Dimension

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$$\dim = \lim_{r \to 0} \log_2 \frac{N_r}{N_{2r}}$$

But finite computers have issues with infinite limits.

Box-Counting Dimension

The formula I actually use:

$$\dim = \frac{1}{2} \log_2 \frac{N_{2r_0}}{N_{8r_0}}$$

$$r_0 = \text{pixel size}$$

Box-Counting Dimension

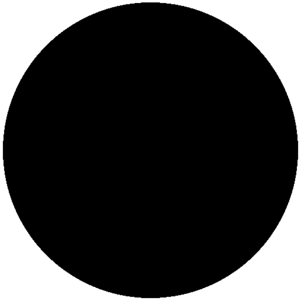
The formula I actually use:

$$\dim = \frac{1}{2} \log_2 \frac{N_{2r_0}}{N_{8r_0}}$$
$$r_0 = \text{pixel size}$$

More on the trade-offs involved later...

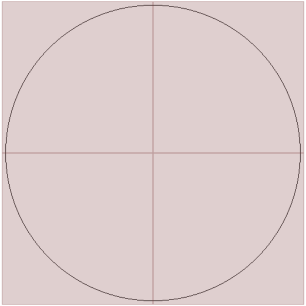
Examples

Example: circle



circle

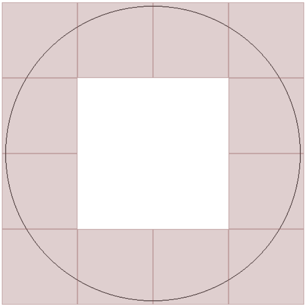
Example: circle



$r = 2^{-1}$

$N = 4$

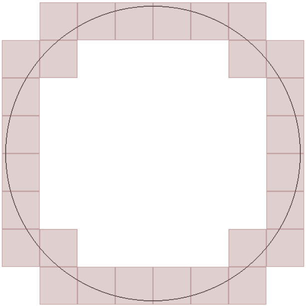
Example: circle



$r = 2^{-2}$

$N = 12$

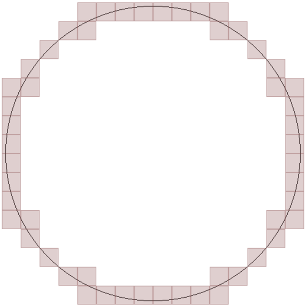
Example: circle



$r = 2^{-3}$

$N = 28$

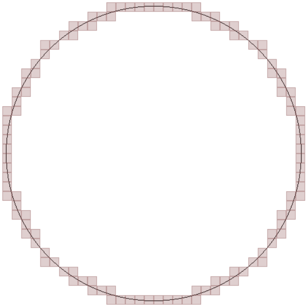
Example: circle



$r = 2^{-4}$

$N = 52$

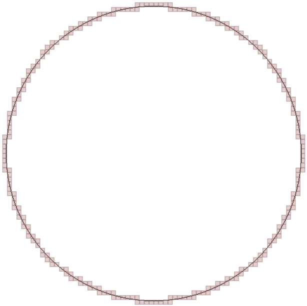
Example: circle



$r = 2^{-5}$

$N = 116$

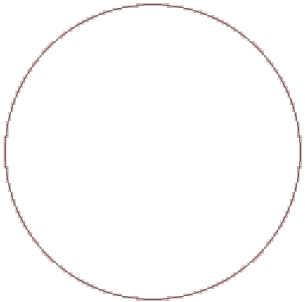
Example: circle



$r = 2^{-6}$

$N = 244$

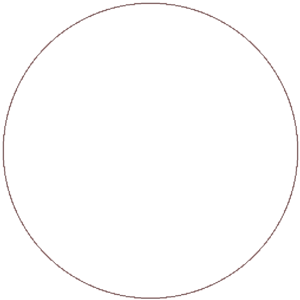
Example: circle



$r = 2^{-7}$

$N = 444$

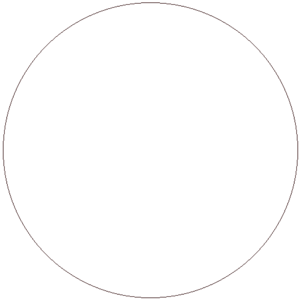
Example: circle



$r = 2^{-8}$

$N = 860$

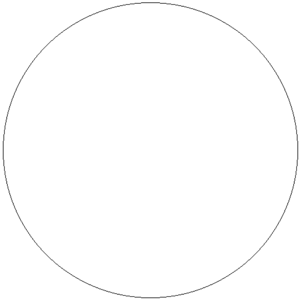
Example: circle



$r = 2^{-9}$

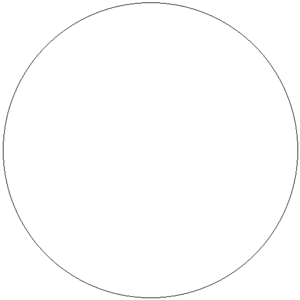
$N = 1412$

Example: circle



$$\dim \approx 0.968\ldots$$

Example: circle



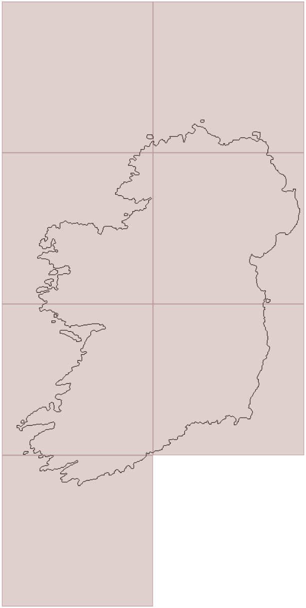
$\dim = 1$

(limit as $r \to 0$)

Example: Ireland

Ireland

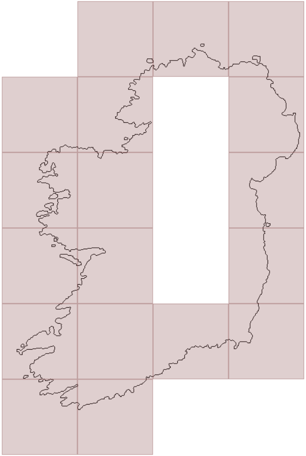
Example: Ireland



$r = 2^{-1}$

$N = 7$

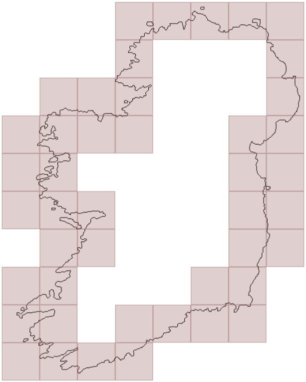
Example: Ireland



$r = 2^{-2}$

$N = 18$

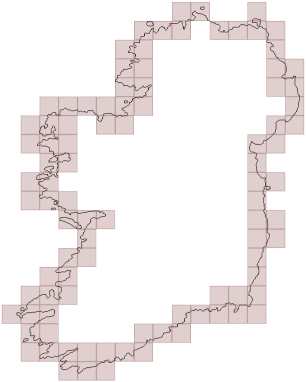
Example: Ireland



$r = 2^{-3}$

$N = 44$

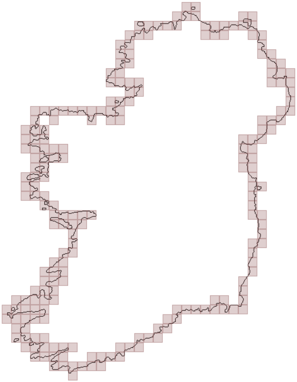
Example: Ireland



$r = 2^{-4}$

$N = 94$

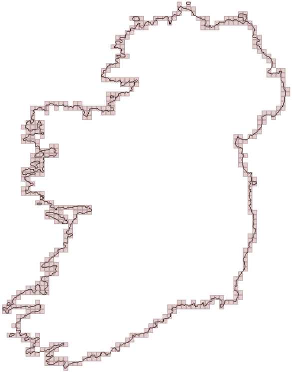
Example: Ireland



$r = 2^{-5}$

$N = 217$

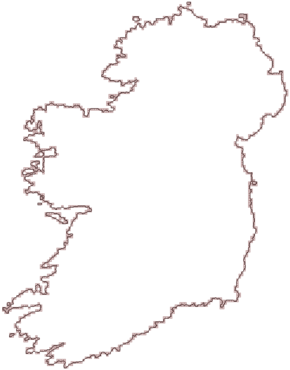
Example: Ireland



$r = 2^{-6}$

$N = 485$

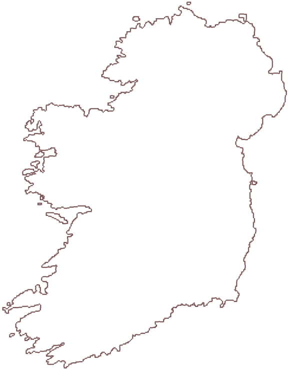
Example: Ireland



$r = 2^{-7}$

$N = 1033$

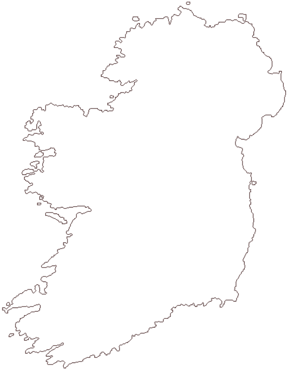
Example: Ireland



$r = 2^{-8}$

$N = 2021$

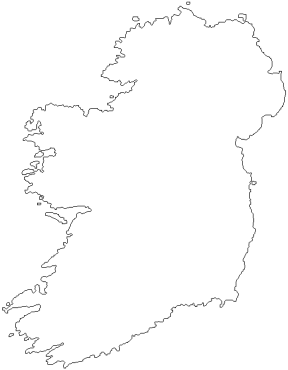
Example: Ireland



$r = 2^{-9}$

$N = 3520$

Example: Ireland

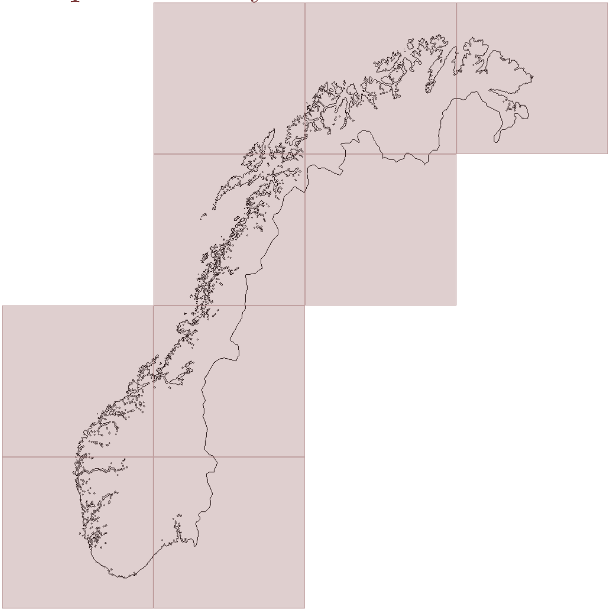


$\dim \approx 1.125\ldots$

Example: Norway

Norway

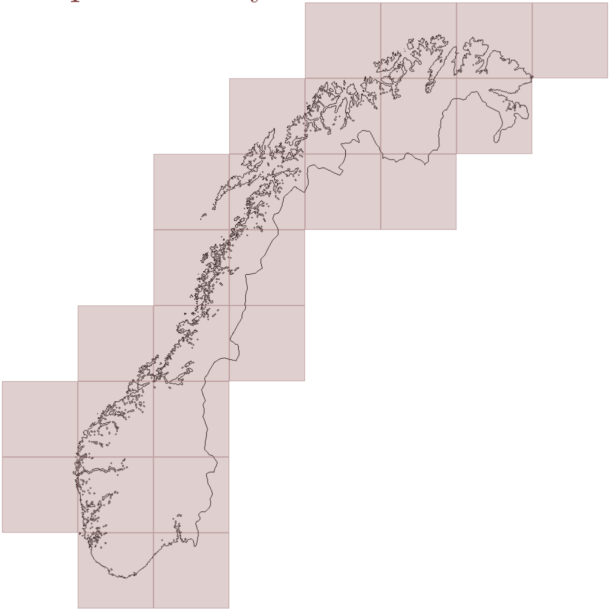
Example: Norway



$r = 2^{-1}$

$N = 9$

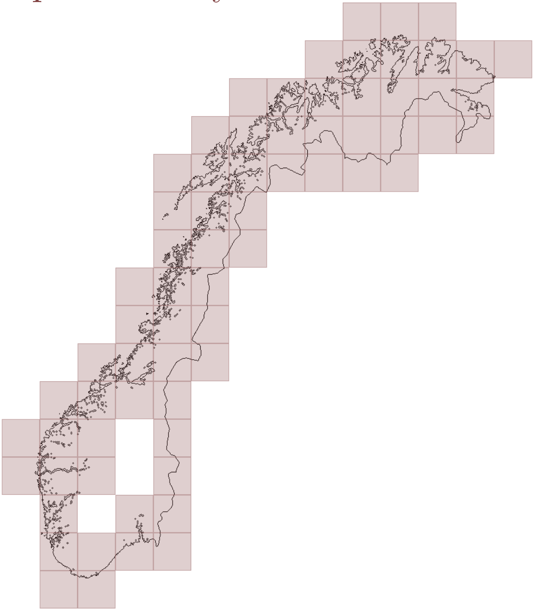
Example: Norway



$r = 2^{-2}$

$N = 25$

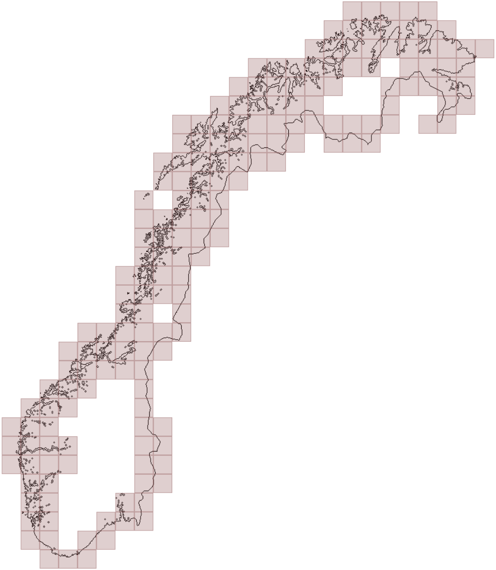
Example: Norway



$r = 2^{-3}$

$N = 68$

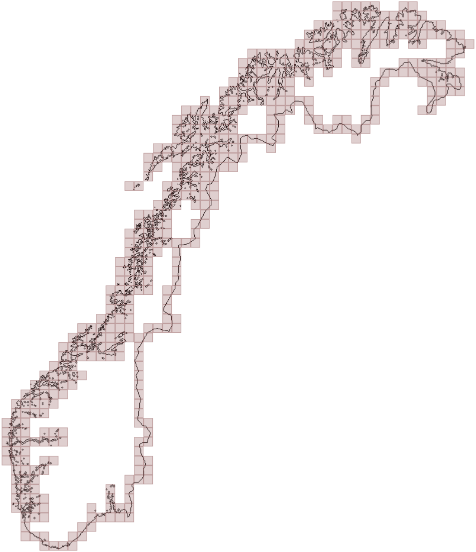
Example: Norway



$r = 2^{-4}$

$N = 180$

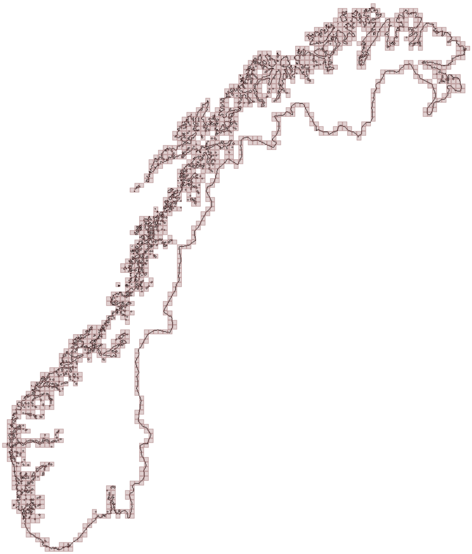
Example: Norway



$r = 2^{-5}$

$N = 488$

Example: Norway



$r = 2^{-6}$

$N = 1310$

Example: Norway

$r = 2^{-7}$

$N = 3333$

Example: Norway

$r = 2^{-8}$

$N = 7641$

Example: Norway

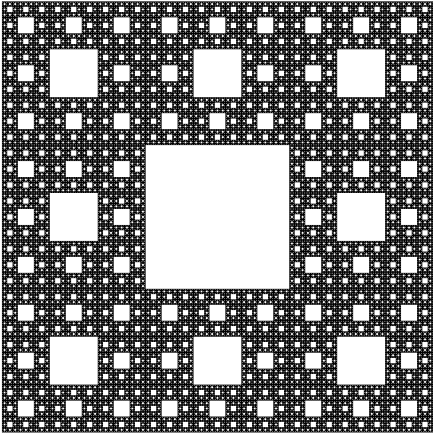
$r = 2^{-9}$

$N = 13070$

Example: Norway

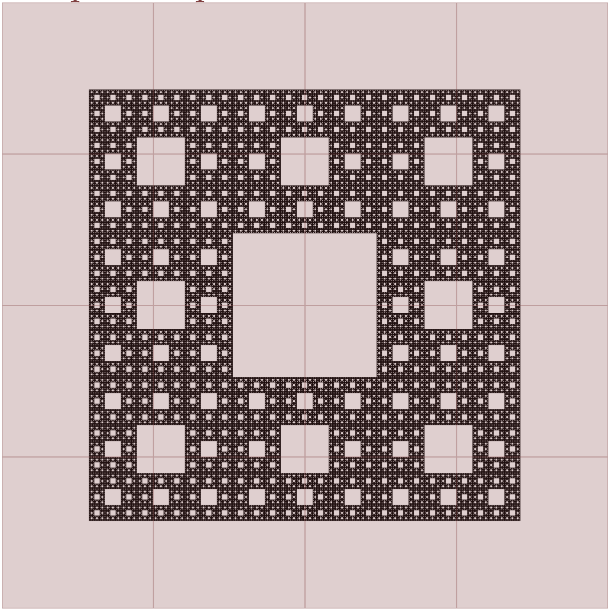
$$\dim \approx 1.385\ldots$$

Example: carpet



carpet

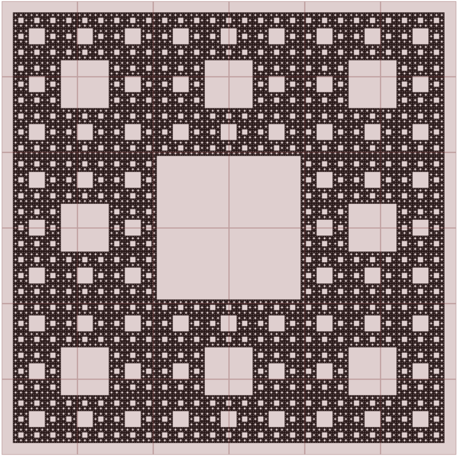
Example: carpet



$r = 2^{-1}$

$N = 16$

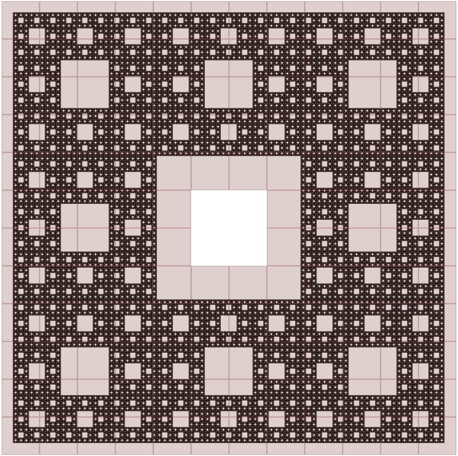
Example: carpet



$r = 2^{-2}$

$N = 36$

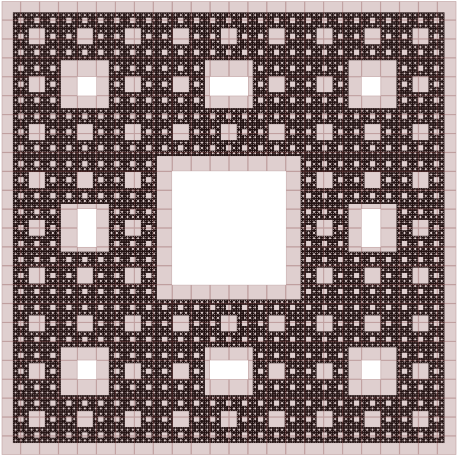
Example: carpet



$r = 2^{-3}$

$N = 140$

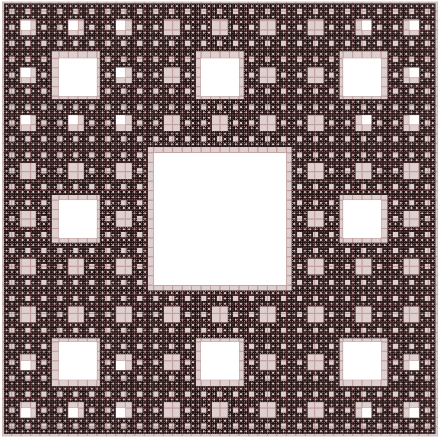
Example: carpet



$$r = 2^{-4}$$

$$N = 528$$

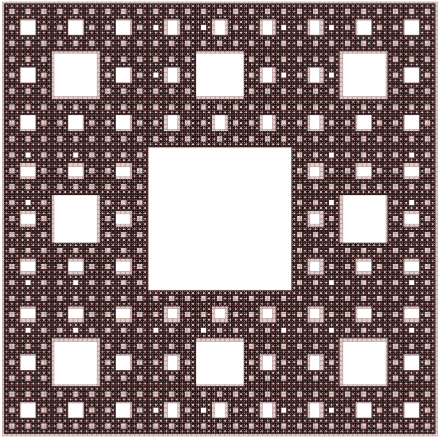
Example: carpet



$r = 2^{-5}$

$N = 1771$

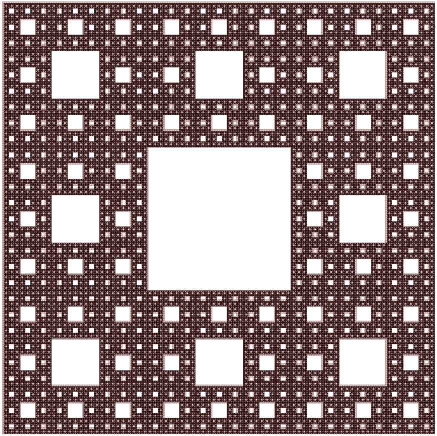
Example: carpet



$r = 2^{-6}$

$N = 6418$

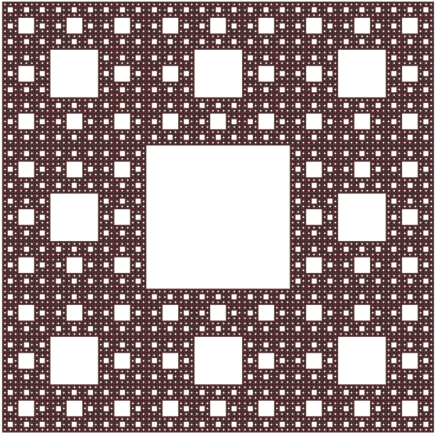
Example: carpet



$r = 2^{-7}$

$N = 23340$

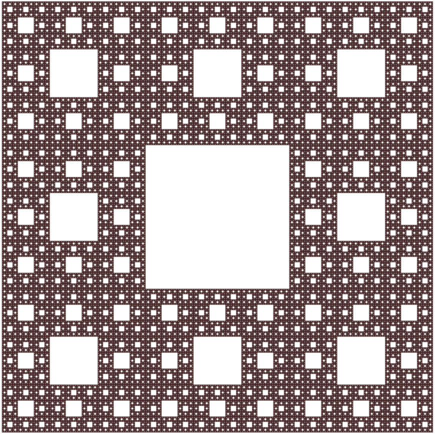
Example: carpet



$r = 2^{-8}$

$N = 82680$

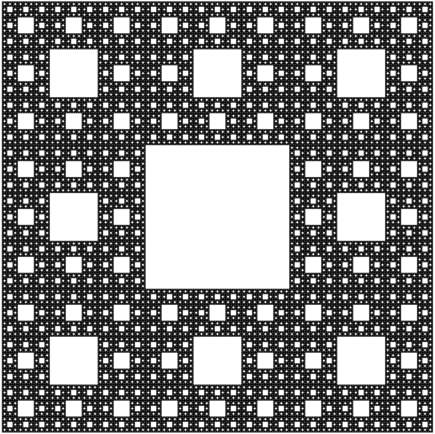
Example: carpet



$r = 2^{-9}$

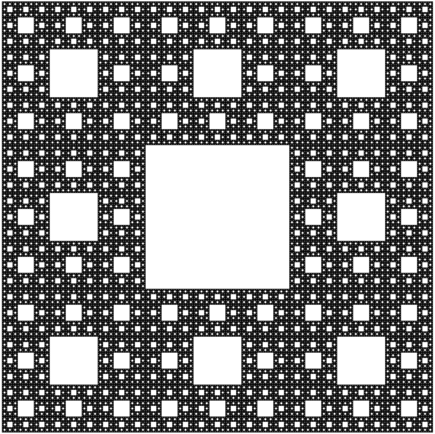
$N = 262144$

Example: carpet



$\dim \approx 1.860\ldots$

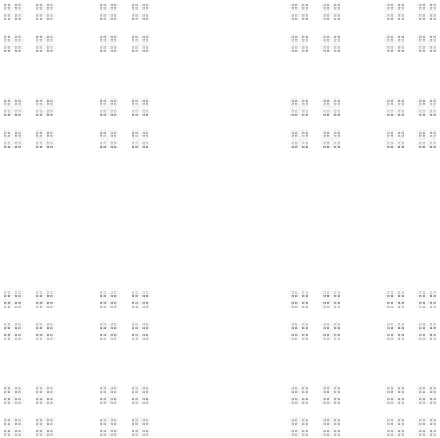
Example: carpet



$$\dim = \frac{\log 8}{\log 3}$$
$$\approx 1.893\ldots$$

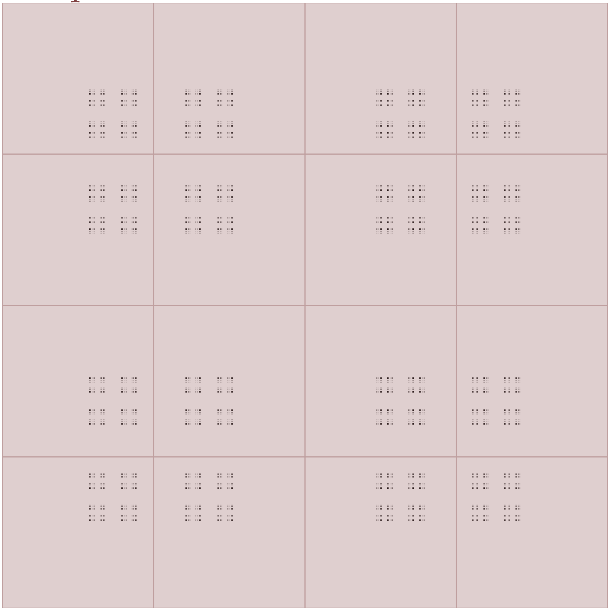
(limit as $r \to 0$)

Example: dust



dust

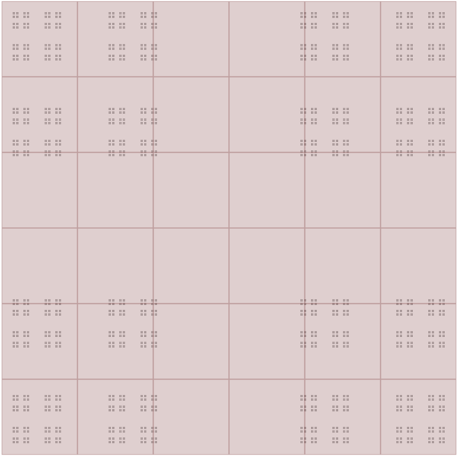
Example: dust



$r = 2^{-1}$

$N = 16$

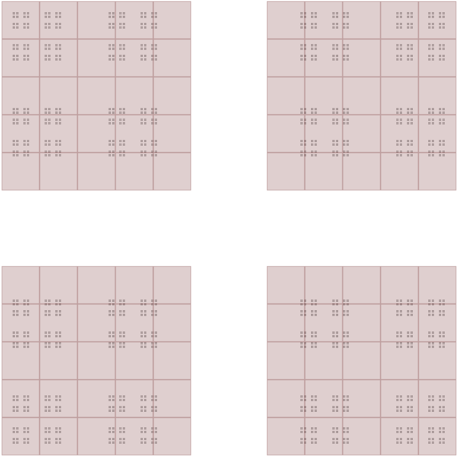
Example: dust



$r = 2^{-2}$

$N = 36$

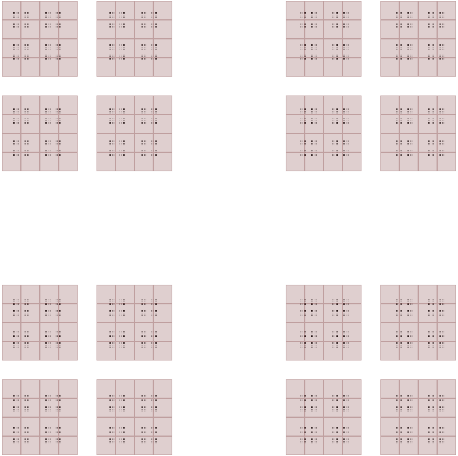
Example: dust



$$r = 2^{-3}$$

$$N = 100$$

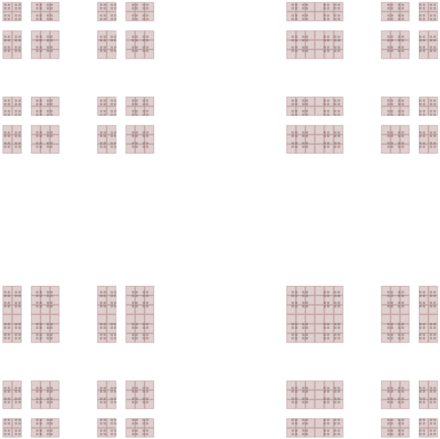
Example: dust



$$r = 2^{-4}$$

$$N = 256$$

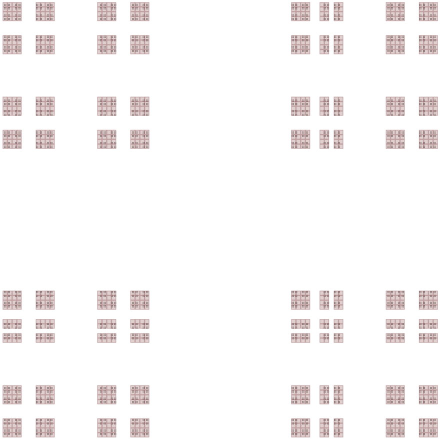
Example: dust



$$r = 2^{-5}$$

$$N = 441$$

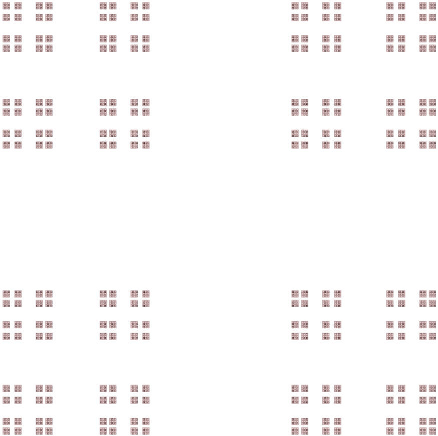
Example: dust



$r = 2^{-6}$

$N = 1024$

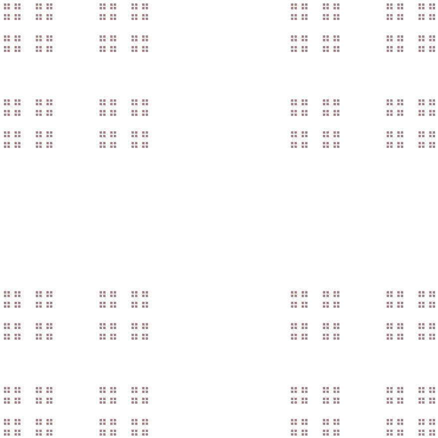
Example: dust



$r = 2^{-7}$

$N = 2304$

Example: dust



$r = 2^{-8}$

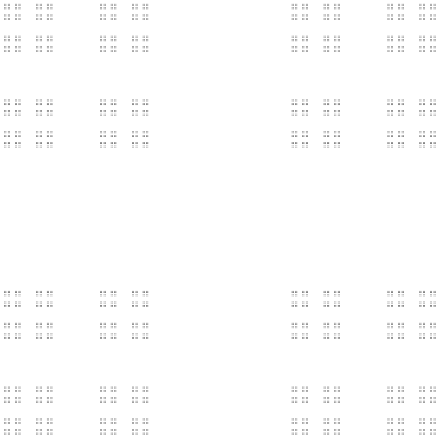
$N = 4096$

Example: dust

$r = 2^{-9}$

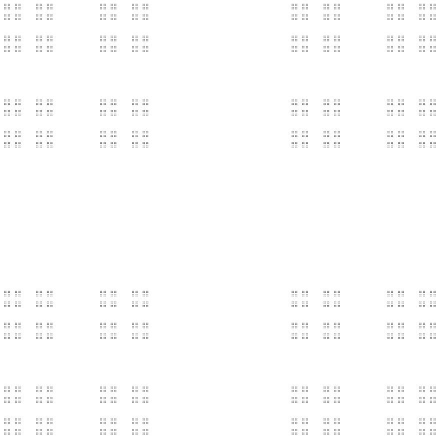
$N = 4096$

Example: dust



$\dim \approx 1.192\ldots$

Example: dust



$$\dim = \frac{\log 4}{\log 3} \approx 1.262\ldots$$

(limit as $r \to 0$)

Julia Sets

Complex Dynamics

Complex Dynamics

Consider the quadratic polynomial:

$$f_c(z) = z^2 + c$$

Complex Dynamics

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$$f_c(z) = z^2 + c$$

Here $z, c \in \mathbb{C}$, complex numbers.

Complex Dynamics

The quadratic polynomial can be iterated:

$$f_c^n = \underbrace{f_c(f_c(\ldots(f_c(f_c}_{n \text{ times}}(z)))\ldots))$$

Complex Dynamics

The quadratic polynomial can be iterated:

$$f_c^n = \underbrace{f_c(f_c(\ldots(f_c(f_c}_{n \text{ times}}(z)))\ldots))$$

Or in more manageable notation:

$$\begin{aligned} f_c^0 \quad (z) &= \qquad z \\ f_c^{n+1}(z) &= f_c^n(f_c\,(z)) \end{aligned}$$

Complex Dynamics

What is the behaviour of $f_c^n(z)$ as $n \to \infty$?

Complex Dynamics

When $c = -2$ there are 2 distinct cases:

Complex Dynamics

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- $f_c^n(z) \to \infty$ as $n \to \infty$

Complex Dynamics

When $c = -2$ there are 2 distinct cases:

- $f_c^n(z) \to \infty$ as $n \to \infty$
- none of the above

When $c = 0$ there are 3 distinct cases:

Complex Dynamics

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Complex Dynamics

When $c = 0$ there are 3 distinct cases:

- $f_c^n(z) \to \infty$ as $n \to \infty$
- $f_c^n(z) \to 0$ as $n \to \infty$

Complex Dynamics

When $c = 0$ there are 3 distinct cases:

- $f_c^n(z) \to \infty$ as $n \to \infty$
- $f_c^n(z) \to 0$ as $n \to \infty$
- none of the above

Complex Dynamics

When $c = 0$ there are 3 distinct cases:

- $f_c^n(z) \to \infty$ as $n \to \infty$
- $f_c^n(z) \to 0$ as $n \to \infty$
- none of the above

- $|z| > 1$

Complex Dynamics

When $c = 0$ there are 3 distinct cases:

- $f_c^n(z) \to \infty$ as $n \to \infty$
- $f_c^n(z) \to 0$ as $n \to \infty$
- none of the above

- $|z| > 1$
- $|z| < 1$

Complex Dynamics

When $c = 0$ there are 3 distinct cases:

- $f_c^n(z) \to \infty$ as $n \to \infty$
- $f_c^n(z) \to 0$ as $n \to \infty$
- none of the above

- $|z| > 1$
- $|z| < 1$
- $|z| = 1$

When $c = -1$ there are 4 distinct cases:

Complex Dynamics

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- Moreover, nearby points stay nearby under iteration.

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- Within each Fatou component the behaviour is the same.
- Moreover, nearby points stay nearby under iteration.
- (In fact, they usually get closer.)

The last case (none of the above) is the Julia set $J(f_c)$.

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- But they stay within the Julia set.

Complex Dynamics

The last case (none of the above) is the Julia set $J(f_c)$.

- Nearby points get further apart under iteration.
- But they stay within the Julia set.
- The Julia set is the boundary of the Fatou components.

Image Generation

Image Generation

What do Julia sets look like?

Image Generation

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The algorithm for determining p from c is quite involved, so I won't go into it now.

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- Define $z_{*0} = \lim_{n\to\infty} f_c^{pn}(0)$.
- z_{*0} satisfies $f_c^p(z_{*0}) = z_{*0}$.
- The solution can be found using Newton's method.

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- If $|f_c^n(z) - z_{*0}| < e$, then $z \in F_{n \bmod p}$.

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- If $|f_c^n(z)| > E$, then $z \in F_{-1}$.
- If $|f_c^n(z) - z_{*0}| < e$, then $z \in F_{n \mod p}$.
- If $n > N$, where N is a maximum iteration count necessary for finite computers, then we give up, and don't know much about z.

Image Generation

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- $\frac{\partial}{\partial z} f_c^{n+1}(z) = 2 f_c^n(z) \frac{\partial}{\partial z} f_c^n(z)$
- In imperative programming language pseudo-code:

```
z := pixel coordinates
d := 1
for n := 0 to N
  d := 2 * z * d
  z := z * z + c
```

Image Generation

Why do we need the derivative?

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Why do we need the derivative?

- The Julia set is the boundary of the Fatou components.
- How to detect the boundary, if there is only one Fatou component?
- Even if there are more components, the boundary might be very thin in places.

Image Generation

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This formula is only valid for F_{-1} with $f_c^n(z) \to \infty$, so it's best to make E as large as reasonably possible.
(There are other formulae for the other cases, but I couldn't get them to work reliably.)

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This image has black pixels near the Julia set, and white pixels elsewhere.

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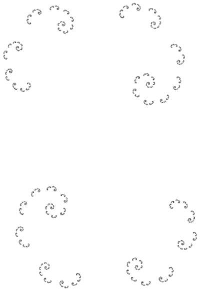
This image has black pixels near the Julia set, and white pixels elsewhere.
An extension is to colour pixels by the value of m, instead of white, as in the following examples.

Examples

Example: needle dust

$c = -2.1$
$+ 0.0i$
$\dim \approx 0.800\ldots$

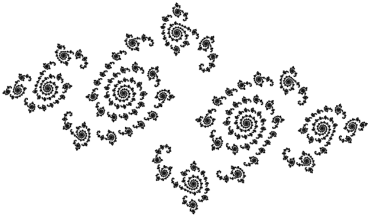
Example: elephant dust



$$c = +0.5 \\ +0.1i$$

$$\dim \approx 1.167\ldots$$

Example: seahorse dust



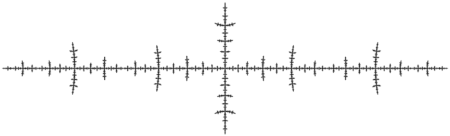
$$c = -0.75 + 0.25i$$

$$\dim \approx 1.609\ldots$$

Example: needle tip dendrite

$$c = -2$$
$$+ 0i$$
$$\dim \approx 0.999\ldots$$

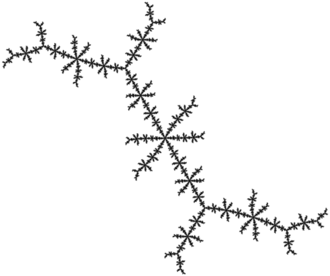
Example: 2-way hub dendrite



$$c = -1.54368 + 0i$$

$\dim \approx 1.450\ldots$

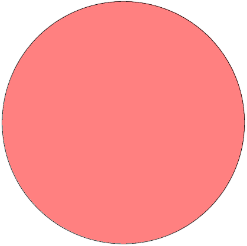
Example: 3-way hub dendrite



$$c = -0.10109 + 0.95628i$$

$\dim \approx 1.564\ldots$

Example: period 1

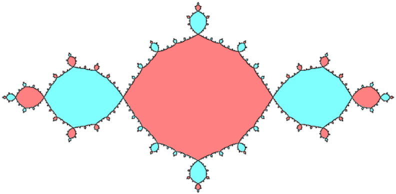


$c = +0$

$+0i$

$\dim \approx 1.086\ldots$

Example: period 2

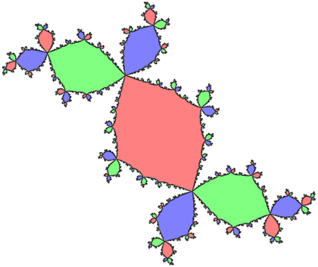


$c = -1$

$+ 0i$

dim $\approx 1.331\ldots$

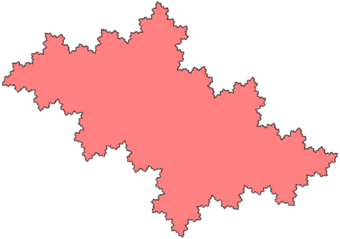
Example: period 3



$c = -0.12256$
$+0.74486i$

$\dim \approx 1.443\ldots$

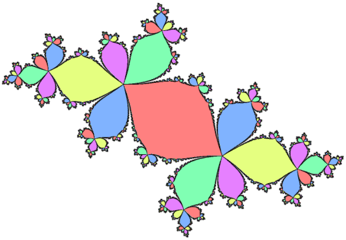
Example: period 1 near 2 over 5



$$c = -0.45400 + 0.49378i$$

$$\dim \approx 1.257\ldots$$

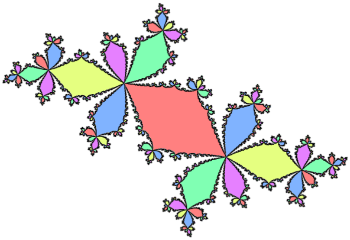
Example: period 5 near 2 over 5



$c = -0.48734$
$+0.53932i$

dim $\approx 1.485\ldots$

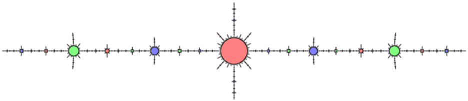
Example: period 5



$$c = -0.50434 + 0.56276i$$

$\dim \approx 1.550\ldots$

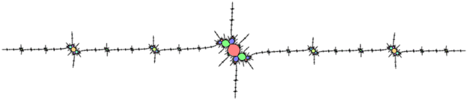
Example: period 3 island



$c = -1.75487$
$+ 0i$

$\dim \approx 1.309\ldots$

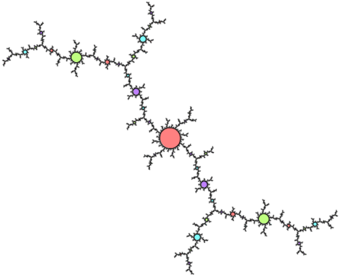
Example: period 3 island 1 over 3 bulb



$$c = -1.75778 + 0.01379i$$

dim $\approx 1.455\ldots$

Example: period 4 island



$$c = -0.15652 + 1.03224i$$

$$\dim \approx 1.463\ldots$$

Fractal Dimension of Julia Sets

Concept

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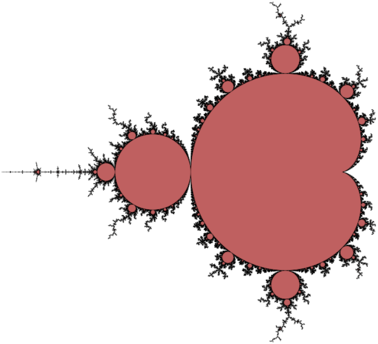
The Mandelbrot set is a c-plane plot of whether $J(f_c)$ is connected.

Concept

The Mandelbrot set is a c-plane plot of whether $J(f_c)$ is connected. Some visualisations of the Mandelbrot set use psychedelic colours, but the mathematical object is binary.

Concept



The Mandelbrot set

Concept

What would a c-plane plot of $\dim J(f_c)$ look like?

Concept

What would a c-plane plot of $\dim J(f_c)$ look like?
$0 \leq \dim J(f_c) \leq 2$, so a spectrum of colours would be necessary.

Implementation

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- C99 supports complex numbers.
- Low-level and efficient.
- Most libraries have C interfaces.

Implementation

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Implementation

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- OpenGL supports programmable graphics hardware (GPUs).
- GPUs are very good at parallel number-crunching tasks, like rendering a Julia set.
- Mipmap generation reduces images to progressively coarser pixel resolutions.
- Occlusion queries can be used for counting pixels.

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- to post-process the Fatou component index and distance estimate into a black and white image of the Julia set;
- to discard pixels below a threshold.

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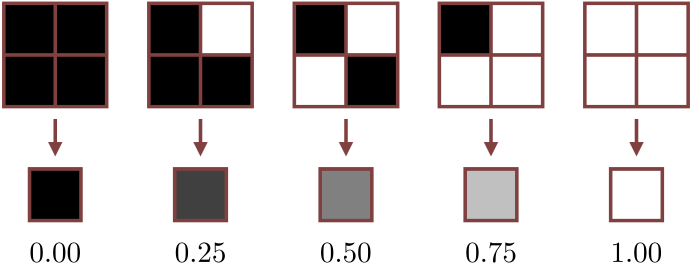

Implementation

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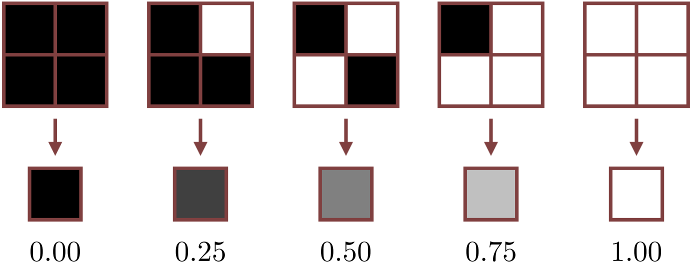

Box-counting should count if **any** subpixel was black.

Implementation

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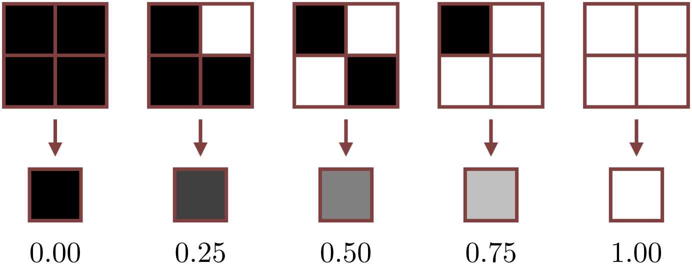

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The solution is to threshold the grey level in each mipmap level.

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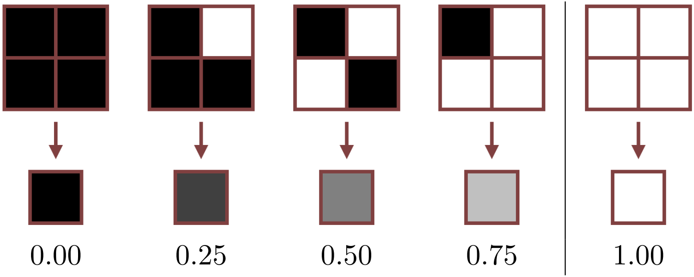

Box-counting should count if **any** subpixel was black.

The solution is to threshold the grey level in each mipmap level.

The threshold should between the lightest grey and white.

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Box-counting is performed with occlusion queries:

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- First clear the depth buffer.

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Box-counting is performed with occlusion queries:

- First clear the depth buffer.
- Then draw the Julia set, discarding pixels below a threshold.
- Then draw again, but further away.
- The depth test prevents pixels that were rendered the first time from being drawn, so only the previously discarded pixels pass.
- The occlusion query counts the number of passed pixels in the second draw.

Implementation

Performance:

Implementation

Performance:

- Faster than a CPU-based implementation.

Implementation

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- But it's still time-consuming.

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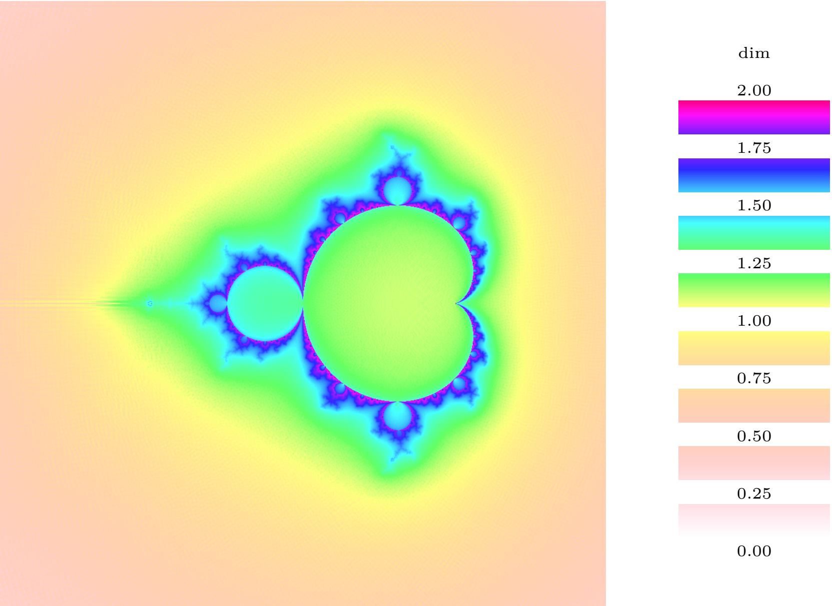
Implementation

Performance:

- Faster than a CPU-based implementation.
- But it's still time-consuming.
- The final image took over 5 hours to render.
- Watching the image appear pixel-by-pixel brings back memories of rendering fractals a couple of decades ago…

Results

Results



Results

But is it accurate?

Results

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No.

Results

But is it accurate?
No.
But it's pretty close.

Results

Recall the formula I actually used:

$$\dim = \frac{1}{2}\log_2 \frac{N_{2r_0}}{N_{8r_0}}$$

$$r_0 = \text{pixel size}$$

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This formula is based on simple linear regression of $\log N$ against $\log r$.

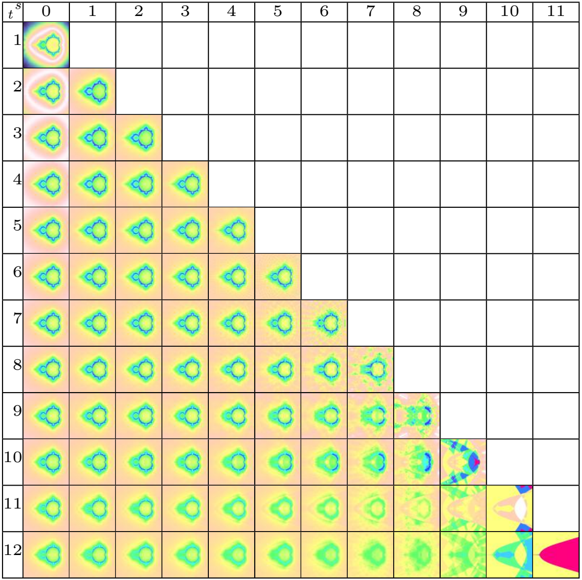
Results

Recall the formula I actually used:

$$\dim = \frac{1}{2} \log_2 \frac{N_{2r_0}}{N_{8r_0}}$$
$$r_0 = \text{pixel size}$$

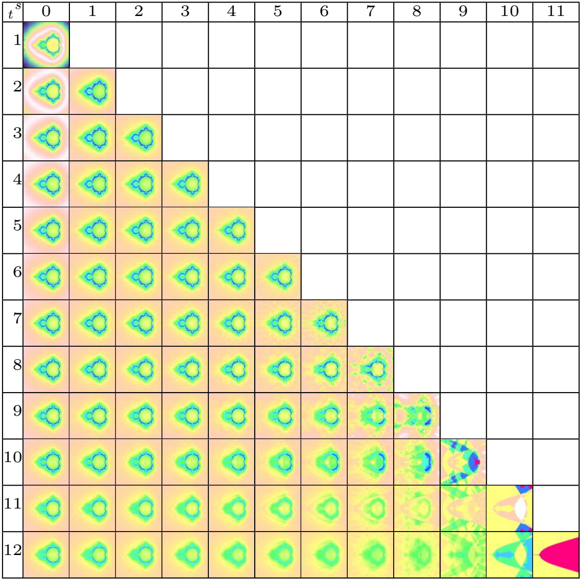
This formula is based on simple linear regression of $\log N$ against $\log r$. I tried all possibilities of $0 \leq s < t \leq 12$ for a regression range between $2^s r_0$ and $2^t r_0$.

Results



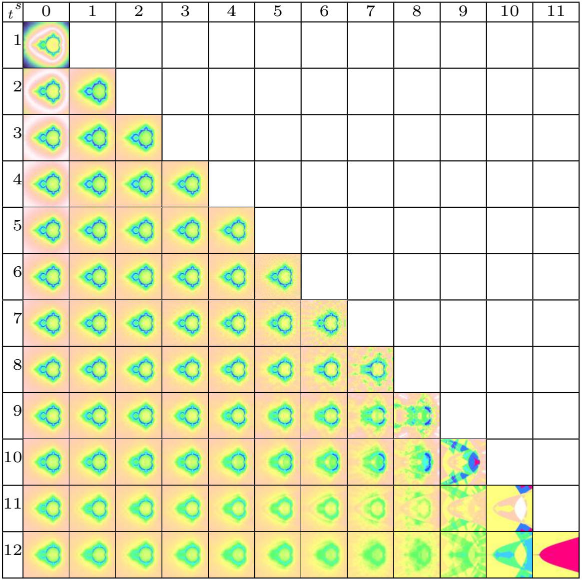
When $s = 0$ and t is small, the dimension calculated is wrong because the Julia set is too inexact at the resolution of the pixel grid.

Results



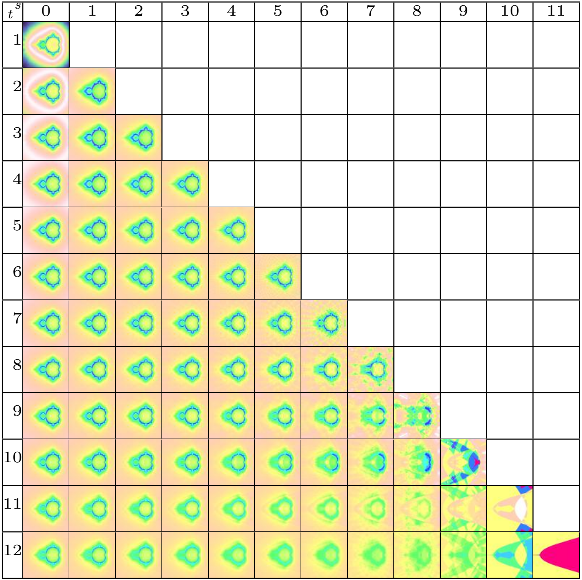
Increasing s a little reduces this artifact of pixel resolution, but t needs to stay small or the results go bad again.

Results



When both s and t are large, the results are nonsense.

Results



The best trade-off seems to be at $s = 1$ and $t = 3$, which gives the formula I actually used.

The End

The End.