Mandelbrot Notebook

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1 Newton's Method: Nucleus

The nucleus n of a hyperbolic component of period p satisfies

$$F^p(0, n) = 0$$

Applying Newton's method in one complex variable gives

$$n_{m+1} = n_m - \frac{F^p(0, n_m)}{\frac{\partial}{\partial c} F^p(0, n_m)}$$

2 Newton's Method: Wucleus

A wucleus w of a point c within a hyperbolic component of period p satisfies

$$F^p(w, c) = w$$

Applying Newton's method in one complex variable gives

$$w_{m+1} = w_m - \frac{F^p(w_m, c) - w_m}{\frac{\partial}{\partial z} F^p(w_m, c) - 1}$$

This iteration is unstable because there are p distinct w_i. Perhaps use p-dimensional Newton's method to find all w_i at once, or trace rays from the orbit of the nucleus to the orbit containing the wuclei.

3 Newton's Method: Bond

The bond b between a hyperbolic component of period p and nucleus n at internal angle θ measured in turns satisfies

$$F^p(w, b) = w$$
$$\frac{\partial}{\partial z} F^p(w, b) = e^{2\pi i\theta}$$

Applying Newton's method in two complex variables gives

$$\begin{pmatrix} \frac{\partial}{\partial z} F^p(w_m, b_m) - 1 & \frac{\partial}{\partial c} F^p(w_m, b_m) \\ \frac{\partial}{\partial z}\frac{\partial}{\partial z} F^p(w_m, b_m) & \frac{\partial}{\partial c}\frac{\partial}{\partial z} F^p(w_m, b_m) \end{pmatrix} \begin{pmatrix} w_{m+1} - w_m \\ b_{m+1} - b_m \end{pmatrix} = -\begin{pmatrix} F^p(w_m, b_m) - w_m \\ \frac{\partial}{\partial z} F^p(w_m, b_m) - e^{2\pi i\theta} \end{pmatrix}$$

4 Newton's Method: Ray In

The next point r along an external ray with current doubled angle θ, current depth p and current radius R satisfies

$$F^p(0, r) = \lambda R e^{2\pi i \theta}$$

where $\lambda < 1$ controls the sharpness of the ray. Applying Newton's method in one complex variable gives

$$r_{m+1} = r_m - \frac{F^p(0, r_m) - \lambda R e^{2\pi i \theta}}{\frac{\partial}{\partial c} F^p(0, r_m)}$$

When crossing dwell bands, double θ and increment p, resetting the radius R. Stop tracing when close to the target (for example when within the basin of attraction for Newton's method for nucleus).

5 Newton's Method: Ray Out

When crossing dwell bands, compute the doubling preimages θ_-, θ_+ of θ

$$\theta_- = \frac{\theta}{2}$$
$$\theta_+ = \frac{\theta + 1}{2}$$

Then compute r_- and r_+ using Newton's method, and choose the nearest to r_m to be r_{m+1}. Collect a list of which choices were made to determine the final θ in high precision without relying on the bits of θ itself. $\lambda > 1$ controls the sharpness of the ray. Stop traccing when r reaches the escape radius.

6 Newton's Method: Preperiodic

A preperiodic point u with period p and preperiod k satisfies

$$F^{p+k}(0, u) = F^k(0, u)$$

Applying Newton's method in one complex variable gives

$$u_{m+1} = u_m - \frac{F^{p+k}(0, u_m) - F^k(0, u_m)}{\frac{\partial}{\partial c} F^{p+k}(0, u_m) - \frac{\partial}{\partial c} F^k(0, u_m)}$$

However this may converge to a u' with lower preperiod. Verify by finding the lowest k' such that Newton's method with preperiod k' starting from u' still converges to u' and check that $k' = k$.

7 Derivatives Calculation

The quadratic polynomial $F(z,c) = z^2 + c$ and its derivatives under iteration

$$
\begin{aligned}
F^{m+1}(z,c) &= F(F^m(z,c),c) \\
\frac{\partial}{\partial z}F^{m+1} &= 2F^m \frac{\partial}{\partial z}F^m \\
\frac{\partial}{\partial z}\frac{\partial}{\partial z}F^{m+1} &= 2F^m \frac{\partial}{\partial z}\frac{\partial}{\partial z}F^m + 2\left(\frac{\partial}{\partial z}F^m\right)^2 \\
\frac{\partial}{\partial c}F^{m+1} &= 2F^m \frac{\partial}{\partial c}F^m + 1 \\
\frac{\partial}{\partial c}\frac{\partial}{\partial z}F^{m+1} &= 2F^m \frac{\partial}{\partial c}\frac{\partial}{\partial z}F^m + 2\frac{\partial}{\partial c}F^m \frac{\partial}{\partial z}F^m
\end{aligned}
$$

with initial values

$$
\begin{aligned}
F^0(z,c) &= z \\
\frac{\partial}{\partial z}F^0 &= 1 \\
\frac{\partial}{\partial z}\frac{\partial}{\partial z}F^0 &= 0 \\
\frac{\partial}{\partial c}F^0 &= 0 \\
\frac{\partial}{\partial c}\frac{\partial}{\partial z}F^0 &= 0
\end{aligned}
$$

8 External Angles: Bulb

The $\frac{p}{q}$ bulb of the period 1 cardioid has external angles

$$.(b_0 b_1 \ldots b_{q-3} 01) .(b_0 b_1 \ldots b_{q-3} 10)$$

where

$$[b_0 b_1 \ldots] = \text{map}(\in (1 - \frac{p}{q}, 1))(\text{iterate}(+\frac{p}{q})\frac{p}{q})$$

9 External Angles: Hub

The $\frac{p}{q}$ bulb has external angles $.(s_-)$ and $.(s_+)$. The junction point of its hub has external angles in increasing order

$$
\begin{aligned}
&.s_-(s_+) \\
&.s_-(\sigma^\beta s_+) \\
&\vdots \\
&.s_-(\sigma^{(q-p-1)\beta} s_+) \\
&.s_+(\sigma^{(q-p)\beta} s_+) \\
&\vdots \\
&.s_+(s_-)
\end{aligned}
$$

where $\frac{p}{q}$ has Farey parents

$$\frac{\alpha}{\beta} < \frac{\gamma}{\delta}$$

and σ is the shift operator

$$\sigma^k(b_0 b_1 \ldots) = b_k b_{k+1} \ldots$$

10 Farey Numbers

Given $\frac{p}{q}$ in lowest terms, it has Farey parents $\frac{\alpha}{\beta} < \frac{\gamma}{\delta}$ where

$$\begin{aligned}\frac{p}{q} &= \frac{\alpha+\gamma}{\beta+\delta} \\ 1 &= p\beta - q\alpha \\ -1 &= p\delta - q\gamma\end{aligned}$$

Parents can be found by recursively searching from $\frac{0}{1}$ and $\frac{1}{1}$.

11 External Angles: Tuning

Given an external angle pair corresponding to a hyperbolic component

$$\begin{aligned}s_- &= .(a_0 a_1 \ldots) \\ s_+ &= .(b_0 b_1 \ldots)\end{aligned}$$

and an external angle

$$t = .c_0 c_1 \ldots (d_0 d_1 \ldots)$$

then t tuned by s is formed by replacing every 0 in t by s_- and every 1 in t by s_+ as finite blocks being the repeated part of each.

12 External Angles: Tips

Given hub angles $a_0, a_1, \ldots a_{m-1}$ the tip of the spoke between a_i and a_{i+1} has external angle with binary representation the longest matching prefix of a_i and a_{i+1} with 1 appended. The widest spoke has the shortest binary representation.

13 Translating Hubs Towards Tips

Translating a hub towards the tip of the widest spoke repeats the last preperiodic digit. Translating towards the tip of any other spoke replaces the preperiodic part with a modified tip: the final 1 becomes 01 for angles below the widest tip and the final 1 becomes 10 for angles above the widest tip.

14 Islands In The Spokes

The largest (lowest period) island after a hub is in the widest spoke. Its external angles are the repetitition of the first p (its period) digits of the external angles of the widest spoke.

Numbering spokes in decreasing order of width with the widest spoke numbered 1, the angled internal address for the path

$$\frac{p}{q} * s_0 s_1 \ldots s_m$$

is

$$1 \frac{p}{q} (q + \sum_{i=0}^{0} s_i)(q + \sum_{i=0}^{1} s_i) \ldots (q + \sum_{i=0}^{m} s_i)$$

15 Islands In The Hairs

An island with period $p > 1$ is surrounded by hairs. There are 2^n lowest period islands in the 2^n hairs at depth n and offset k, each with period

$$q = np + k$$

Their external angles are formed by counting in binary using the external angles of the parent period p island as digits

$$e = .(\pm_0 \pm_1 \pm_2 \ldots \pm_{n-1} 0) \text{ for } k = 1$$

where the external angles of p are

$$+ = .(+_0 +_1 \ldots +_{p-1})$$
$$- = .(-_0 -_1 \ldots -_{p-1})$$

16 Islands In Embedded Julias

An island of period q in the hairs of an island of period p is surrounded by an embedded Julia set. If the external angles of p are $.(+)$ and $.(-)$ and an external angle of q is $.(\ldots 0)$ then the external angles heading outwards along the hair within the embedded Julia set are

$$.(\ldots 0[+])$$

and heading inwards along the hair are

$$.(\ldots 0[-])$$

Mixing inwards and outwards motions gives

TODO

17 Multiply Embedded Julias

TODO diagram

18 Continuous Iteration Count

If n is the lowest such that

$$|z_n| > R >> 2$$

then

$$v = n - \log_2 \frac{\log |z_n|}{\log R}$$

where the value subtracted is in $[0, 1)$. When combined with the final angle $\theta = \arg z_n = \tan^{-1} \frac{\Im(z_n)}{\Re(z_n)}$ these can be used to tile the exterior with hyperbolic squares.

TODO diagram

The local coordinates of x are

$$(v \mod 1, \frac{\theta}{2\pi} \mod 1)$$

19 Interior Coordinates

The interior coordinate b for a point c within a hyperbolic component of period p can be found by applying Newton's method in one complex variable to find a wucleus w for c satisfying

$$F^p(w, c) = w$$

then

$$b = \frac{\partial}{\partial z} F^p(w, c)$$

The interior coordinate satisfies

$$|b| < 1$$

and can be used to map a disc into the interior of a component.

20 Exterior Distance

Given c outside the Mandelbrot set, the exterior distance estimate

$$d = \lim_{n \to \infty} 2 \frac{|F^n(0, c)| \log |F^n(0, c)|}{|\frac{\partial}{\partial c} F^n(0, c)|}$$

satisfies by the Koebe $\frac{1}{4}$ Theorem

$$\forall c'. |c - c'| < \frac{d}{4} \implies c' \notin M$$

Shading using $t = \tanh \frac{d}{\text{pixel size}}$ works well because $t \in [0, 1)$ and $\tanh 4 \approx 1$.

21 Interior Distance

Given c inside a hyperbolic component of period p, with a wucleus w, then the interior distance estimate

$$d = \frac{1 - |\frac{\partial}{\partial z} F^p(w,c)|^2}{\left| \frac{\partial}{\partial c}\frac{\partial}{\partial z} F^p(w,c) + \frac{\frac{\partial}{\partial z}\frac{\partial}{\partial z} F^p(w,c) \frac{\partial}{\partial c} F^p(w,c)}{1 - \frac{\partial}{\partial z} F^p(w,c)} \right|}$$

satisfies by the Koebe $\frac{1}{4}$ Theorem

$$\forall c'. |c - c'| < \frac{d}{4} \implies c' \in M$$

Shading using $t = \tanh \frac{d}{\text{pixel size}}$ works well because $t \in [0, 1)$ and $\tanh 4 \approx 1$.

22 Perturbation

TODO

23 Finding Atoms

Given an angled internal address, the nucleus and size of an atom can be found by:

- splitting the address to island and children
- finding the external angles of the island
- tracing the external ray towards the island
- using Newton's method to find the nucleus of the island

and then iteratively for the children

- find the bond at the internal angle of the child
- using the size estimate and normal at the bond point to estimate the nucleus
- using Newton's method to find the nucleus of the child

24 Child Size Estimates

For the $\frac{q}{p}$ child of a cardioid of size R:

$$r \approx \frac{R}{p^2} \sin\left(2\pi \frac{q}{p}\right)$$

For the $\frac{q}{p}$ child of a disc of size R:

$$r \approx \frac{R}{p^2}$$

The size of a cardioid is approximated by the distance from its cusp to its $\frac{1}{2}$ bond, and for a disc its radius. The nucleus n of the child at internal angle θ with bond b is approximated by

$$n \approx b - ir\frac{\partial\theta}{\partial b}$$

For the main period 1 cardioid

$$b = u(1-u)$$

where

$$u = \frac{e^{2\pi i \frac{q}{p}}}{2}$$

25 Atom Shape Estimates

Given the nucleus c and period p of a hyperbolic component, the shape discriminant ϵ_p satisfies

$$\epsilon_p = -\frac{1}{(\frac{\partial}{\partial c})(\frac{\partial}{\partial z})}\left(\frac{\frac{\partial}{\partial c}\frac{\partial}{\partial c}}{2\frac{\partial}{\partial c}} + \frac{\frac{\partial}{\partial c}\frac{\partial}{\partial z}}{\frac{\partial}{\partial z}}\right)$$

where the derivatives are evaluated at $F^p(c,c)$. Then $\epsilon_p \approx 0$ for cardioids and $\epsilon_p \approx 1$ for discs.

26 Atom Size Estimates

Given nucleus c and period p, calculate γ by

$$\begin{aligned} z_0 &= 0 \\ z_{m+1} &= z_m^2 + c \\ \lambda &= \prod_{i=0}^{p-1} 2z_i \\ \beta &= \sum_{i=0}^{p-1} \frac{1}{\prod_{j=1}^{i} 2z_i} \\ \gamma &= \frac{1}{\beta\lambda^2} \end{aligned}$$

then the size estimate is $|\gamma|$ and the orientation estimate is $\arg\gamma$.

27 Tracing Equipotentials

Proceed as external ray tracing, but keep dwell and radius fixed and increment the angle by $\frac{2\pi}{\text{sharpness}}$.

28 Buddhabrot

For each exterior c, colour hue according to dwell and plot the z iterates into an accumulation buffer. Post-process this high dynamic range image to map brightness to point density.

29 Atom Domain Size Estimate

Given nucleus c with period p, find $1 \leq q < p$ such that $|F^q(0,c)|$ is minimized. Then the atom domain size estimate is

$$R = \frac{F^q(0,c)}{\frac{\partial}{\partial c} F^p(0,c)}$$

30 Stretching Cusps

Moebius transformation:

$$f(z) = \frac{az+b}{cz+d}$$
$$f^{-1}(z) = \frac{dz-b}{-cz+a}$$

Unwrapping circle to line:

$$g(z) = \frac{(z-p_0)(p_1-p_\infty)}{(z-p_\infty)(p_1-p_0)}$$

Mapping cardioids to circles:

$$h(z) = \sqrt{1-4z} - 1$$
$$h^{-1}(z) = \frac{1-(z+1)^2}{4}$$

Derivatives for distance estimation:

$$\frac{\partial}{\partial z} \frac{az+b}{cz+d} = \frac{ad-bc}{(cz+d)^2}$$
$$\frac{\partial}{\partial z} \frac{1-(z+1)^2}{4} = -\frac{z+1}{2}$$