Perturbation techniques applied to the Mandelbrot set

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1 The Mandelbrot set

Consider iterations of a quadratic polynomial F over complex numbers:

$$\begin{aligned} F(z,c) &= z^2 + c \\ F^{n+1}(z,c) &= F(F^n(z,c),c) \end{aligned} \tag{1}$$

The Mandelbrot set M is the set of c values for which the iterates of $z = 0$ remain bounded:

$$M = \{c \in \mathbb{C} : F^n(0,c) \nrightarrow \infty \text{ as } n \to \infty\} \tag{2}$$

The shape of M is exceedingly intricate, with variety increasing the further we zoom in. Zooming in requires more precise numerical methods: we need at least enough information to distinguish nearby points in the region we want to visualize. Using this much information for each point is certainly good enough, but as the precision increases it gets slower.

2 Perturbation techniques

The idea[1] is simple: take a high precision orbit for one point as a reference, and assuming a well-behaved function, the orbits for points near to the reference will for the most part be near to the reference orbit. Instead of computing nearby orbits at high precision, save time and effort by computing only the difference from the reference orbit.

This works out because if you have two high precision numbers close together, their difference has less meaningful precision. For example

$$\begin{aligned} A &= 123456798 \\ B &= 123456789 \\ A - B &= 9 \end{aligned} \tag{3}$$

even with A and B known to 9 significant figures, we can only determine their difference to 1 significant figure. Most commonly computers use binary floating point arithmetic, and the precision lost e when subtracting nearby values x and y can be measured[2] in bits:

$$e = -\log_2\left(1 - \frac{\min(|x|,|y|)}{\max(|x|,|y|)}\right) \tag{4}$$

Measuring the error that accumulates when subtracting (or adding numbers with opposite sign) while calculating the perturbed orbit can inform us when the reference orbit isn't good enough (possibly giving us a badly glitched blobby image), and even provide hints for choosing a better reference location.

3 Applying the technique

Choose a reference c and iterate z (and its derivatives if needed) with high precision:

$$
\begin{aligned}
z_0 &= z & z_{n+1} &= z_n^2 + c \\
\frac{\partial}{\partial z} z_0 &= 1 & \frac{\partial}{\partial z} z_{n+1} &= 2 z_n \frac{\partial}{\partial z} z_n \\
\frac{\partial}{\partial c} z_0 &= 0 & \frac{\partial}{\partial c} z_{n+1} &= 2 z_n \frac{\partial}{\partial c} z_n + 1 \\
\frac{\partial}{\partial z} \frac{\partial}{\partial z} z_0 &= 0 & \frac{\partial}{\partial z} \frac{\partial}{\partial z} z_{n+1} &= 2 z_n \frac{\partial}{\partial z} \frac{\partial}{\partial z} z_n + 2 {\frac{\partial}{\partial z} z_n}^2 \\
\frac{\partial}{\partial c} \frac{\partial}{\partial z} z_0 &= 0 & \frac{\partial}{\partial c} \frac{\partial}{\partial z} z_{n+1} &= 2 z_n \frac{\partial}{\partial c} \frac{\partial}{\partial z} z_n + 2 \frac{\partial}{\partial c} z_n \frac{\partial}{\partial z} z_n
\end{aligned}
\tag{5}
$$

Define the deltas $\langle\langle c \rangle\rangle, \langle\langle z \rangle\rangle, \ldots$ for a nearby orbit $C, Z, \ldots$ which can be computed with lower precision:

$$
\begin{aligned}
C &= c + \langle\langle c \rangle\rangle & Z_n &= z_n + \langle\langle z_n \rangle\rangle \\
\frac{\partial}{\partial z} Z_n &= \frac{\partial}{\partial z} z_n + \langle\langle \frac{\partial}{\partial z} z_n \rangle\rangle & \frac{\partial}{\partial c} Z_n &= \frac{\partial}{\partial c} z_n + \langle\langle \frac{\partial}{\partial c} z_n \rangle\rangle \\
\frac{\partial}{\partial z} \frac{\partial}{\partial z} Z_n &= \frac{\partial}{\partial z} \frac{\partial}{\partial z} z_n + \langle\langle \frac{\partial}{\partial z} \frac{\partial}{\partial z} z_n \rangle\rangle & \frac{\partial}{\partial c} \frac{\partial}{\partial z} Z_n &= \frac{\partial}{\partial c} \frac{\partial}{\partial z} z_n + \langle\langle \frac{\partial}{\partial c} \frac{\partial}{\partial z} z_n \rangle\rangle
\end{aligned}
\tag{6}
$$

Some boring algebraic manipulation gives the iterations for the deltas:

$$
\begin{aligned}
\langle\langle z_{n+1} \rangle\rangle &= 2 z_n \langle\langle z_n \rangle\rangle + \langle\langle z_n \rangle\rangle^2 + \langle\langle c \rangle\rangle \\
\langle\langle \frac{\partial}{\partial z} z_{n+1} \rangle\rangle &= 2 \left(\frac{\partial}{\partial z} z_n \langle\langle z_n \rangle\rangle + z_n \langle\langle \frac{\partial}{\partial z} z_n \rangle\rangle + \langle\langle z_n \rangle\rangle \langle\langle \frac{\partial}{\partial z} z_n \rangle\rangle \right) \\
\langle\langle \frac{\partial}{\partial c} z_{n+1} \rangle\rangle &= 2 \left(\frac{\partial}{\partial c} z_n \langle\langle z_n \rangle\rangle + z_n \langle\langle \frac{\partial}{\partial c} z_n \rangle\rangle + \langle\langle z_n \rangle\rangle \langle\langle \frac{\partial}{\partial c} z_n \rangle\rangle \right) \\
\langle\langle \frac{\partial}{\partial z} \frac{\partial}{\partial z} z_{n+1} \rangle\rangle &= 2 \left(\frac{\partial}{\partial z} \frac{\partial}{\partial z} z_n \langle\langle z_n \rangle\rangle + z_n \langle\langle \frac{\partial}{\partial z} \frac{\partial}{\partial z} z_n \rangle\rangle + 2 z_n \langle\langle \frac{\partial}{\partial z} z_n \rangle\rangle \right. \\
&\qquad \left. + \langle\langle z_n \rangle\rangle \langle\langle \frac{\partial}{\partial z} \frac{\partial}{\partial z} z_n \rangle\rangle + \langle\langle \frac{\partial}{\partial z} z_n \rangle\rangle^2 \right) \\
\langle\langle \frac{\partial}{\partial c} \frac{\partial}{\partial z} z_{n+1} \rangle\rangle &= 2 \left(\frac{\partial}{\partial c} \frac{\partial}{\partial z} z_n \langle\langle z_n \rangle\rangle + z_n \langle\langle \frac{\partial}{\partial c} \frac{\partial}{\partial z} z_n \rangle\rangle + \langle\langle z_n \rangle\rangle \langle\langle \frac{\partial}{\partial c} \frac{\partial}{\partial z} z_n \rangle\rangle \right. \\
&\qquad \left. + \frac{\partial}{\partial c} z_n \langle\langle \frac{\partial}{\partial z} z_n \rangle\rangle + \langle\langle \frac{\partial}{\partial c} z_n \rangle\rangle \frac{\partial}{\partial z} z_n + \langle\langle \frac{\partial}{\partial c} z_n \rangle\rangle \langle\langle \frac{\partial}{\partial z} z_n \rangle\rangle \right)
\end{aligned}
\tag{7}
$$

For interior coordinates[3] and interior distance estimation[4], we need to solve $Z_p = Z_0$, but when $z_p = z_0$ we can apply the perturbation technique to Newton's method for root finding:

$$
\begin{aligned}
Z_0^{(m+1)} &= Z_0^{(m)} - \frac{Z_p^{(m)} - Z_0^{(m)}}{\frac{\partial}{\partial z} Z_p^{(m)} - 1} \\
\langle\langle z_0 \rangle\rangle^{(m+1)} &= \langle\langle z_0 \rangle\rangle^{(m)} - \frac{\langle\langle z_p \rangle\rangle^{(m)} - \langle\langle z_0 \rangle\rangle^{(m)}}{\frac{\partial}{\partial z} z_p + \langle\langle \frac{\partial}{\partial z} z_p \rangle\rangle^{(m)} - 1}
\end{aligned}
\tag{8}
$$

The precondition isn't too onerous, as it's often sensible to choose a periodic point as a reference, and it's enough if p is a multiple of the period of the reference.

4 Series approximation

The deltas are polynomial series in $\langle\!\langle c \rangle\!\rangle$. Define the coeffecients $[\![z_n]\!]_m$, $[\![\frac{\partial}{\partial c} z_n]\!]_m$ of the polynomials for each delta:

$$\langle\!\langle z_n \rangle\!\rangle = \sum [\![z_n]\!]_m \langle\!\langle c \rangle\!\rangle^m \quad \langle\!\langle \frac{\partial}{\partial c} z_n \rangle\!\rangle = \sum [\![\frac{\partial}{\partial c} z_n]\!]_m \langle\!\langle c \rangle\!\rangle^m \tag{9}$$

Some boring algebraic manipulation gives the iterations for the first few coefficients of $\langle\!\langle z_n \rangle\!\rangle$:

$$\begin{aligned}
[\![z_{n+1}]\!]_1 &= 2 z_n [\![z_n]\!]_1 + 1 \\
[\![z_{n+1}]\!]_2 &= 2 z_n [\![z_n]\!]_2 + [\![z_n]\!]_1^2 \\
[\![z_{n+1}]\!]_3 &= 2 z_n [\![z_n]\!]_3 + 2 [\![z_n]\!]_1 [\![z_n]\!]_2
\end{aligned} \tag{10}$$

and similarly for the coefficients of $\langle\!\langle \frac{\partial}{\partial c} z_n \rangle\!\rangle$:

$$\begin{aligned}
[\![\frac{\partial}{\partial c} z_{n+1}]\!]_1 &= 2 \left(\frac{\partial}{\partial c} z_n [\![z_n]\!]_1 + z_n [\![\frac{\partial}{\partial c} z_n]\!]_1 \right) \\
[\![\frac{\partial}{\partial c} z_{n+1}]\!]_2 &= 2 \left(\frac{\partial}{\partial c} z_n [\![z_n]\!]_2 + z_n [\![\frac{\partial}{\partial c} z_n]\!]_2 + [\![z_n]\!]_1 [\![\frac{\partial}{\partial c} z_n]\!]_1 \right) \\
[\![\frac{\partial}{\partial c} z_{n+1}]\!]_3 &= 2 \left(\frac{\partial}{\partial c} z_n [\![z_n]\!]_3 + z_n [\![\frac{\partial}{\partial c} z_n]\!]_3 + [\![z_n]\!]_1 [\![\frac{\partial}{\partial c} z_n]\!]_2 + [\![z_n]\!]_2 [\![\frac{\partial}{\partial c} z_n]\!]_1 \right)
\end{aligned} \tag{11}$$

The coefficients are independent of $\langle\!\langle c \rangle\!\rangle$ so the same coefficients can be used for many points in an image, and when $|\langle\!\langle c \rangle\!\rangle|$ is small the sum can be approximated by truncating to the first few terms. However, the coefficients grow quickly as n increases, which limits how long the per-reference approximation remains valid, after which we have to switch back to per-point delta iteration.

Notes

1 `http://superfractalthing.co.nf/sft_maths.pdf`

2 `http://en.wikipedia.org/wiki/Loss_of_significance#Loss_of_significant_bits`

3 `http://mathr.co.uk/blog/2013-04-01_interior_coordinates_in_the_mandelbrot_set.html`

4 `http://en.wikipedia.org/wiki/Mandelbrot_set#Interior_distance_estimation`