mandelbla

Mandelbrot set via bivariate linear approximation.

theory

mandelbrot set:

High precision reference orbit:

$$Z_{m+1} = Z_m^2 + C$$

m starts at 0 with $Z_0 = 0$.

perturbation

Low precision deltas relative to high precision orbit. Pixel orbit $Z_m + z_n$, $C + c$.

$$z_{n+1} = 2Z_m z_n + z_n^2 + c$$

m and n start at 0 with $z_0 = 0$.

rebasing

Rebasing to avoid glitches: when

$$|Z_m + z_n| < |z_n|$$

replace z_n with $Z_m + z_n$ and reset the reference iteration count m to 0.

Reference: https://fractalforums.org/fractal-mathematics-and-new-theories/28/another-solution-to-perturbation-glitches/4360/msg29835#msg29835

bivariate linear approximation

When Z is large and z is small, the iterations can be approximated by bivariate linear function;

$$z \to Az + Bc$$

This is valid when

$$|Z + z| >> |z|$$

which implies

$$|z| << (|Z| - |B||c|)/(|A| + 1)$$

by ball interval arithmetic.

single step

$$A = 2Z_m$$
$$B = 1$$
$$R = \max\left\{0, 0.001\frac{|Z_m| - |B||c|}{|A|}\right\}$$

The "fudge factor" ≈ 0.001 seems to work but is not rigourous ...

merging steps

If T_x skips l_x iterations from iteration m_x when $|z| < R_x$ and T_y skips l_y iterations from iteration $m_x + l_x$ when $|z| < R_y$ then $T_z = T_y \circ T_x$ skips $l_x + l_y$ iterations from iteration m_x when $|z| < R_z$:

$$z \to A_y(A_x z + B_x c) + B_y c = A_z z + B_z c$$
$$A_z = A_y A_x$$
$$B_z = A_y B_x + B_y$$
$$R_z = \min\{R_x, \frac{R_y - |B_x||c|}{|A_x|}\}$$

table construction

Suppose the reference has M iterations. Create M BLAs each skipping 1 iteration (this can be done in parallel). Then merge neighbours without overlap to create $\lceil \frac{M}{2} \rceil$ each skipping 2 iterations (except for perhaps the last which skips less). Repeat until there is only 1 BLA skipping $M - 1$ iterations: it's best to start the merge from iteration 1 because reference iteration 0 always corresponds to a non-linear perturbation step as $Z = 0$.

The resulting table as $O(M)$ elements.

table lookup

Find the BLA starting from iteration m that has the largest skip l satisfying $|z| < R$. If there is none, do a perturbation iteration. Check for rebasing opportunities after each BLA application or perturbation step.

error bounds

the above has a conceptual flaw, needing an arbitrary "fudge factor"

remainder bound

Taylor remainder provides rigourous bounds:

$$z_{n+l} = A_{m,l} z_n + B_{m,l} c + D_{m,l} z_n^2 + E_{m,l} z_n c + F_{m,l} c^2 + O(\cdot^3)$$

$$R_{m,l} = \max \left\{ |D_{m,l}|, \frac{|E_{m,l}|}{2}, |F_{m,l}| \right\}$$

$$z_{n+l} = A_{m,l} z_n + B_{m,l} c + a_{z_{n,m,l}}$$

$$\left| a_{z_{n,m,l}} \right| < R_{m,l} (|z_n|^2 + |z_n||c| + |c|^2)$$

$a_{z_{n,m,l}}$ is absolute error in z_{n+l}. Relative error is

$$e_{z_{n,m,l}} = \frac{a_{z_{n,m,l}}}{z_{n+l}}$$

single step

$$\begin{aligned} l &= 1 \\ A &= 2Z_m \\ B &= 1 \\ D &= 1 \\ E &= 0 \\ F &= 0 \end{aligned}$$

merging steps

$$\begin{aligned} m_y &= m_x + l_x \\ m_{y \circ x} &= m_x \\ l_{y \circ x} &= l_x + l_y \\ A_{y \circ x} &= A_y A_x \\ B_{y \circ x} &= A_y B_x + B_y \\ D_{y \circ x} &= A_y D_x + D_y A_x^2 \\ E_{y \circ x} &= A_y E_x + 2 D_y A_x B_x + E_y A_x \\ F_{y \circ x} &= A_y F_x + D_y B_x^2 + E_y B_x + F_y \end{aligned}$$

backward error

Suppose e_c is absolute error in c. For an accurate image, we want bounds on z such that e_z, the absolute error introduced into z by the bilinear approximation, corresponds to $|e_c| < k$, where k is the pixel spacing. As we will be chaining multiple transformations, better to scale k by a small number so that $|e_c| < k$ still holds after all of them: doing L transformations means $|e_c| < \frac{k}{L}$ is certainly good enough.

The absolute error in z is the remainder bound multiplied by $O(\cdot^2)$:

$$e_z = \max\left\{|D|, \frac{|E|}{2}, |F|\right\}(|z|^2 + |z||c| + |c|^2)$$

The forward error from a change in c is Be_c, so we want

$$|B||e_c| = \frac{|B|k}{L} > |e_z| = \max\left\{|D|, \frac{|E|}{2}, |F|\right\}(|z|^2 + |z||c| + |c|^2)$$

so

$$|z|^2 + |c||z| + \left(|c|^2 - \frac{|B|k}{L\max\left\{|D|, \frac{|E|}{2}, |F|\right\}}\right) < 0$$

which is a quadratic in $|z|$ so can be solved; we need to minimize the result for all c in the view of radius $r_c = k\frac{\sqrt{w^2+h^2}}{2}$, which gives:

$$|z| < \min_{|c|<r_c}\left\{\frac{-|c| + \sqrt{|c|^2 - 4(|c|^2 - \frac{|B|k}{L\max\left\{|D|, \frac{|E|}{2}, |F|\right\}})}}{2}\right\}$$

In the limit $|c| \to 0$ this simplifies to

$$|z| < \sqrt{\frac{|B|k}{L\max\left\{|D|, \frac{|E|}{2}, |F|\right\}}}$$

implementation

input

- $C :=$ view center (high precision)
- $M :=$ reference count limit, can be lower than N if periodic
- $N :=$ iteration count limit
- h, $w :=$ image size in pixels
- $k :=$ pixel spacing ($k = 4/(h \times \text{Zoom factor})$)

reference calculation

```
Z[M] low precision array
Z := 0 (high precision)
m := 0
while |Z| < 2 && m < M
  Z[m] := round(Z)
  Z := Z^2 + c
  m := m + 1
```

table data

```
struct TableItem
{
  l : int
  r : real
  A, B : complex
}
dmax := O(log_2(M))
Table[dmax] : ptr TableItem
```

table initialization

```
Table[0] := allocate(M - 1, TableItem)
parallel for m := 1 to M - 1
  A := 2 Z[m]
  B := 1
  r := max( 0, |Z[m]| - |B| k h) / (|A| + 1) )
  Table[0][m-1] = { 1, r, A, B }
```

table merging

```
for d := 0 to dmax
  Table[d + 1] := allocate((count(Table[d]) + 1) / 2, TableItem)
  parallel for j := 1 to count(Table[d + 1])
    jx := 2 (j - 1);
    jy := jx + 1;
    if (jy < count(Table[d]))
      x := Table[d][jx]
      y := Table[d][jy]
      A := y.A * x.A
      B := y.A * x.B + y.B
      r := min( x.r, max( 0, (y.r - |x.B| h k) / |x.A| ) )
      Table[d + 1][j - 1] := { l, r, A, B }
    else
      Table[d + 1][j - 1] := Table[d][jx]
  if count(Table[d + 1]) == 1
```

```
    break
```

table lookup

Find `Table[d][j]` with largest `l` such that `m = (j << d) + 1` and `r > |z|`. Result may be `null` if no such `TableItem`.

pixel iteration

```
while
    iteration < Iterations &&
    perturb_iteration < PerturbIterations &&
    |Z+z| < EscapeRadius
  while
      iteration < Iterations &&
      perturb_iteration < PerturbIterations &&
      |Z+z| < EscapeRadius &&
      t := lookup table with reference iteration and |z|
    // bla iteration
    // rebase if required
  // perturbation iteration
  // rebase if required
```

benchmarks

AMD Ryzen 7 2700X Eight-Core Processor desktop with 16 CPU threads. 1920x1080 image resolution.

Dinkydau's "Flake"

- mandelbla: 0.5s
- kf-2.15.5~git (prerelease): 2.6s
- kf-2.15.4: 5.1s

Dinkydau's "Evolution of Trees"

- mandelbla: 2.3s
- kf-2.15.5~git (prerelease): 37s
- kf-2.15.4: 1m21s

Fractal Universe "Hard Location"

- mandelbla: 39s
- kf-2.15.5~git (prerelease): not attempted
- kf-2.15.4: not attempted